Supplemental Matrix Theory Problems

In Problems 1-3: Solve the hint problem, and then do the following:

(a) Find the characteristic polynomial and the eigenvalues of S.

(b) Find an orthonormal basis of each eigenspace of S.

(c) Find an orthogonal diagonalization: Q orthogonal and Λ diagonal such that $S = Q\Lambda Q^T$.

1. $S = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$. Hint problem: find a vector v such that $S + 1 = vv^T$.

2. $S = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 0 \end{pmatrix}$. Hint problem: find a constant c and a vector v such that $S + c = vv^T$.

3. Let $S = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \\ 1 & 0 & 5 & -2 \\ 0 & 1 & -2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$.

Hint problem: compute $\mathrm{Null}(S)$. Show that the conjugates of S by DP and PD are both $6 - S$. Use S conjugate to $6 - S$ to see that if λ is an eigenvalue of S, then $6 - \lambda$ is too.

4. Let Z be a non-invertible symmetric 2×2 matrix with trace -1.

(a) Find its characteristic polynomial $\mathrm{char}_Z(t)$.

(b) Find Z^2 in terms of Z.

(c) Assume that $(2, 1)^T$ is an eigenvector of Z. Find all possibilities for Z.

In Problems 5-6: fix $n > 2$, and let x and y be non-zero orthogonal elements of $\mathbb{R}^n$. Let $\hat{x}$ and $\hat{y}$ be the unit vectors along x and y, i.e., $\hat{x} = x/|x|$ and $\hat{y} = y/|y|$.

5. Consider the matrix $B = xx^T + yy^T$.

(a) In terms of x and y, describe an orthonormal eigenbasis for B.

(b) Find Λ diagonal and Q orthogonal such that $B = Q\Lambda Q^T$. Describe Q.

(c) Find the characteristic polynomial $\mathrm{char}_B(t)$.

6. Repeat Problem **??** for the matrix $K = xy^T + yx^T$. Hint: consider $\frac{1}{\sqrt{2}}(\hat{x} \pm \hat{y})$.

7. Find the PDS square root of $S = \frac{1}{13}\begin{pmatrix} 40 & -18 \\ -18 & 25 \end{pmatrix}$.

8. Find the PSDS square root of $S = \begin{pmatrix} 5 & -3\sqrt{2} & 5 \\ -3\sqrt{2} & 10 & -3\sqrt{2} \\ 5 & -3\sqrt{2} & 5 \end{pmatrix}$.

9. Find the Cholesky decomposition $C^T C$ of $S = \begin{pmatrix} 9 & 3 & -3 \\ 3 & 5 & 1 \\ -3 & 1 & 3 \end{pmatrix}$.

10. Graph $5x_1^2 - 6x_1x_2 + 5x_2^2 = 8$. Your plot should be neatly and carefully drawn, and large: at least a third of a page. Indicate the directions and lengths of the major and minor axes.

11. The matrix $E = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$ has eigenvalues 2 and 8. Set $\Lambda = \begin{pmatrix} 2 & \\ & 8 \end{pmatrix}$.

(a) Find all orthogonal matrices M such that $E = M\Lambda M^T$.

(b) Describe the action of E on $\mathbb{R}^2$.

(c) For each M, describe its action on $\mathbb{R}^2$. Give its eigenvectors and eigenvalues. If it rotates $\mathbb{R}^2$, give the angle and direction. If it reflects $\mathbb{R}^2$, give the mirror line.

12. Here are two orthogonal matrices. One is a rotation, and one is a reflection:

$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}, \qquad G = \frac{1}{2}\begin{pmatrix} 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}.$$

(a) Decide which is which.

(b) Find the angle, the axis, and the "right hand rule" direction of the rotation.

(c) Find the axis of the reflection.

(d) Let $u = \frac{1}{2}(-1, 1, \sqrt{2})^T$. Compute $I - 2uu^T$. Explain the connection.

Definitions for reference: • ON means orthonormal.

- PDS means positive definite symmetric and PSDS means positive semidefinite symmetric.
- If S is any PSDS matrix, $S^{1/2}$ denotes its PSDS square root.
- For A any $m \times n$ matrix, $A^T A$ is PSDS. Its *modulus* $|A|$ is defined to be $(A^T A)^{1/2}$.
- Let A be any square matrix. A *polar decomposition* is a factorization $A = Y|A|$, where Y is orthogonal. If A is invertible, then Y is unique.

13. Let $L = \begin{pmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{pmatrix}$.

(a) Find the modulus $|L|$ of L.

(b) Find the polar decomposition $Y|L|$ of L.

(c) Find the singular value decomposition of L.

14. Let $R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(a) Find the modulus $|R|$ of R.

(b) Find two polar decompositions $Y|R|$ of R.

(c) Find two corresponding singular value decompositions of R.

15. Let $N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(a) Find a reduced SVD and a full SVD.

(b) Find the Schmidt decomposition.

(c) Find N^+, N^+N, and NN^+.

(d) For what unit vectors v is the length of Nv maximal?

16. Repeat Problem 15 for $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$.

17. Repeat Problem 15 for $C = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}$. Hint: the σ_i are small integers.

18. Here is an SVD:

$$\begin{pmatrix} 7 & 1 \\ 5 & 5 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \sqrt{10} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \times \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

(a) Use it to give the reduced SVDs of the following three matrices:

$$\begin{pmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 7 & 0 & 1 \\ 5 & 0 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 7 & 0 & 1 \\ 0 & 0 & 0 \\ 5 & 0 & 5 \end{pmatrix}.$$

(b) Find also A^+, AA^+, and A^+A as A runs through these three matrices.

19. Fix non-zero vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Consider the rank 1 $m \times n$ matrix $Z = xy^T$.

(a) Find the factors U_r, Σ_r, and V_r^T of a reduced SVD of Z.

(b) Find the Schmidt decomposition of Z.

(c) Find the pseudo-inverse Z^+.

(d) Compute ZZ^+ and Z^+Z.

20. Let X be an $n \times n$ diagonal matrix with $X_{11} < X_{22} < \cdots < X_{nn} < 0$.

(a) Find its singular values.

(b) Describe all of its SVDs.

21. Let Y be an $n \times n$ orthogonal matrix.

(a) Find its singular values.

(b) Describe what happens as you go through the steps to produce its SVD.

(c) At what steps do you have to make arbitrary choices?

(d) Give all possible SVDs of Y.

22. Let A be an $m \times n$ matrix of rank n.

(a) Prove that $|A|$ is invertible.

(b) Let $Y = A|A|^{-1}$. Prove that the columns of Y are an ON basis of $\mathcal{R}(A)$.

(c) Let $V\Sigma V^T$ be an orthogonal diagonalization of $|A|$. In terms of Y and V, find U_r such that $U_\mathrm{r}\Sigma V^T$ is a reduced SVD of A.

23. Let A be an $m \times n$ matrix with reduced SVD and Schmidt decomposition

$$A = U_\mathrm{r}\Sigma_\mathrm{r}V_\mathrm{r}^T = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T.$$

(a) Give the reduced SVDs of $(A^+)^T$ and $(A^T)^+$ in terms of U_r, Σ_r, and V_r.

(b) Show that pseudo-inversion and transposition commute: $(A^+)^T = (A^T)^+$.

(c) Give the Schmidt decomposition of A^{+T} in terms of that of A.

In Problems 24-29:

(a) Find a basis for each eigenspace V_λ.

(b) Find a basis for each generalized eigenspace V_λ^{g}.

(c) Find X and Γ such that $X\Gamma X^{-1}$ is a block diagonalization.

24. $A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$.

25. $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

26. $C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

27. $D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ & & 1 & 0 \\ & & 1 & 0 \end{pmatrix}$.

28. $E = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$.

29. $F = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

30. Let $G = \begin{pmatrix} 3 & 3 & 0 \\ 1 & 1 & 4 \\ -4 & -4 & -4 \end{pmatrix}$.

(a) Find the eigenspaces of G.

(b) Find λ and r such that $\mathrm{Null}(G - \lambda)^r$ is 2-dimensional.

(c) Find a block diagonalization of G.

31. Find a 3×3 matrix N such that $N^3 = 0$,

$$\mathrm{Ker}(N^2) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}, \qquad \mathrm{Ker}(N) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

32. Let $X\Gamma X^{-1}$ be a block diagonalization. Describe the generalized eigenspaces V_λ^{g} of Γ.

33. Consider the process for finding a block diagonalization $X\Gamma X^{-1}$. Describe the form Γ would have if you were to reverse the order of the basis of $\mathbb{R}^n$ used to form X.