7.7 Improper Integrals

- To find the area under a curve with infinite domain, such as the curve \( y = (\ln x) / x^2 \) from \( x = 1 \) to \( x = \infty \) (Figure 7.12a), we ask, “What is the integral when the domain is slightly less?” and examine the answer as the domain increases to the limit. (Figure 7.14)

- **Definition.** Integrals with infinite limits of integration are called **improper integrals.**

1. If \( f(x) \) is continuous on \([a, \infty)\), then
   \[
   \int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.
   \]

2. If \( f(x) \) is continuous on \((-\infty, b]\), then
   \[
   \int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.
   \]

3. If \( f(x) \) is continuous on \((-\infty, \infty)\), then
   \[
   \int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,
   \]
   where \( c \) is any real number.

- If the limit in parts 1 and 2 is finite, the improper integral **converges** and the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges.** In part 3, the integral on the left-hand side of the equation **converges** if both improper integrals on the right-hand side converge; otherwise it **diverges** and has no value. [32]

- It can be shown that the integral \( \int_1^\infty \frac{dx}{x^p} \) converges to the value \( 1/(p - 1) \) if \( p > 1 \) and it diverges if \( p \leq 1 \).

- **Definition.** Integrals of functions that become infinite at a point within the interval of integration are **improper integrals.**

1. If \( f(x) \) is continuous on \((a, b]\), then
   \[
   \int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.
   \]

2. If \( f(x) \) is continuous on \([a, b)\), then
   \[
   \int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.
   \]

3. If \( f(x) \) is continuous on \([a, c) \cup (c, b]\), then
   \[
   \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.
   \]

   [16,8,68]

- **Theorem 3: Direct Comparison Test.** Let \( f \) and \( g \) be continuous on \([a, \infty)\) with \( 0 \leq f(x) \leq g(x) \) for all \( x \geq a \). Then
1. $\int_1^\infty f(x) \, dx$ converges if $\int_1^\infty g(x) \, dx$ converges.

2. $\int_1^\infty g(x) \, dx$ diverges if $\int_1^\infty f(x) \, dx$ diverges.

- **Theorem 4: Limit Comparison Test.** If the positive functions $f$ and $g$ are continuous on $[a, \infty)$ and if
  \[
  \lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty,
  \]
  then
  \[
  \int_a^\infty f(x) \, dx \quad \text{and} \quad \int_a^\infty g(x) \, dx
  \]
  both converge or both diverge.  

  [Typo: 5th line of the solution to Example 10 on page 595 should read “Therefore, $\int_1^\infty \frac{dx}{1+x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.”]