6.6 Euler’s Method; Population Models

- If we cannot find an exact solution for an initial value problem \( y' = f(x, y), y(x_0) = y_0 \), we can probably use a computer to generate a table of approximate numerical values of \( y \) for values of \( x \) in an appropriate interval. Such a table is called a numerical solution of the problem, and the method by which we generate the table is called a numerical method.

- **Euler’s Method.** Given a differential equation \( \frac{dy}{dx} = f(x, y) \) and an initial condition \( y(x_0) = y_0 \), we can approximate the solution \( y \) by its linearization

\[
L(x) = y(x_0) + y'(x_0)(x - x_0) \quad \text{or} \quad L(x) = y_0 + f(x_0, y_0)(x - x_0).
\]

The function \( L \) gives a good approximation to the solution \( y \) in a short interval about \( x_0 \).

The basis of Euler’s method is to patch together a string of linearizations to approximate the curve over a longer stretch. We know the point \((x_0, y_0)\) lies on the solution curve. Suppose that \( x_1 = x_0 + dx \) where the increment \( dx \) is small, then

\[
y_1 = L(x_1) = y_0 + f(x_0, y_0)\,dx
\]

is a good approximation to the exact solution value \( y(x_1) \). Using the point \((x_1, y_1)\) and the slope \( f(x_1, y_1) \) of the solution curve through \((x_1, y_1)\), we take a second step. Setting \( x_2 = x_1 + dx \), we use the linearization of the solution curve through \((x_1, y_1)\) to calculate

\[
y_2 = y_1 + f(x_1, y_1)\,dx.
\]

This gives the next approximation \((x_2, y_2)\) to values along the solution curve \( y \) (Figure 6.19). Continuing in this fashion, we obtain the next several approximations. [2]

- **Improved Euler’s Method.** We first estimate \( y_n \) as in the original Euler method, but denote it by \( z_n \). We then take the average of \( f(x_{n-1}, y_{n-1}) \) and \( f(x_n, z_n) \) in place of \( f(x_{n-1}, y_{n-1}) \) in the next step. Thus we calculate the next approximation \( y_n \) using

\[
z_n = y_{n-1} + f(x_{n-1}, y_{n-1})\,dx
\]

\[
y_n = y_{n-1} + \left[ \frac{f(x_{n-1}, y_{n-1}) + f(x_n, z_n)}{2} \right] dx.
\]

- Recall that the exponential model for population growth

\[
\frac{dP}{dt} = kP
\]

assumes unlimited growth and assumes that the relative growth rate \( k \) is constant. But if the relative growth rate is positive and decreases as the population increases due to environmental, economic, and other factors, we propose the logistic growth model

\[
\frac{dP}{dt} = r(M - P)P,
\]

where \( M \) is the maximum population, or carrying capacity, that the environment is capable of sustaining in the long run. The graphical solution curves to the logistic model were obtained in Section 3.4 and are displayed in Figure 6.22. [10]