8.3 Infinite Series

- **Definition.** Given a sequence of numbers \( \{ a_n \} \), an expression of the form

\[
a_1 + a_2 + a_3 + \cdots + a_n + \cdots
\]

is an infinite series. The number \( a_n \) is the \textit{nth term} of the series. The partial sums of the series form a sequence

\[
\begin{align*}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= \sum_{k=1}^{n} a_k \\
&\vdots
\end{align*}
\]

of real numbers, each defined as a finite sum. If the sequence of partial sums has a limit \( S \) as \( n \to \infty \), we say that the series converges to the sum \( S \), and we write

\[
a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k = S.
\]

Otherwise, we say that the series diverges.

- **Geometric series** are series of the form

\[
a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}
\]

in which \( a \) and \( r \) are fixed numbers and \( a \neq 0 \). The ratio \( r \) can be positive or negative. If \(|r| \neq 1\), we can show that the \textit{nth} partial sum is

\[
s_n = \frac{a(1 - r^n)}{1 - r}, \quad (r \neq 1).
\]

If \(|r| < 1\), then \( r^n \to 0 \) as \( n \to \infty \) and \( s_n \to a/(1 - r) \). If \(|r| > 1\), then \(|r^n| \to \infty \) and the series diverges.

If \( r = 1 \), the \textit{nth} partial sum of the geometric series is

\[
s_n = a + a(1) + a(1)^2 + \cdots a(1)^{n-1} = na,
\]

and the series diverges. If \( r = -1 \), the series diverges because the \textit{nth} partial sums alternate between \( a \) and 0. [2,8,20,42,6,Example 6]

- **Theorem 6: Limit of the \textit{nth} Term of a Convergent Series.** If \( \sum_{n=1}^{\infty} a_n \) converges, then \( a_n \to 0 \).

  \textbf{Proof.} Let \( S \) represent the series’ sum and \( s_n = a_1 + a_2 + a_3 + \cdots + a_n \) the \textit{nth} partial sum. When \( n \) is large, both \( s_n \) and \( s_{n-1} \) are close to \( S \), so

\[
a_n = s_n - s_{n-1} \to S - S = 0.
\]

- **\textit{nth}-Term Test for Divergence.** \( \sum_{n=1}^{\infty} a_n \) diverges if \( \lim_{n \to \infty} a_n \) fails to exist or is different from zero. [Example 8,9]

- **Theorem 7: Properties of Convergent Series.** If \( \sum a_n = A \) and \( \sum b_n = B \) are convergent series, then

  1. **Sum Rule:** \( \sum (a_n + b_n) = \sum a_n + \sum b_n = A + B \)
  2. **Difference Rule:** \( \sum (a_n - b_n) = \sum a_n - \sum b_n = A - B \)
  3. **Constant Multiple Rule:** \( \sum ka_n = k \sum a_n = kA \) (any number \( k \)). [10]