6.1 Logarithms

- **Definition.** The natural logarithm of a positive number \( x \), written as \( \ln x \), is the value of an integral:

\[
\ln x = \int_{1}^{x} \frac{1}{t} \, dt, \quad x > 0
\]

Note that this is a different approach from starting with \( e^x \) and defining \( \ln x \) as its inverse.

- **Derivative of \( y = \ln x \).** By the first part of the Fundamental Theorem of Calculus (in Section 4.5),

\[
\frac{d}{dx} \ln x = \frac{d}{dx} \int_{1}^{x} \frac{1}{t} \, dt = \frac{1}{x}.
\]

For every positive value of \( x \), therefore,

\[
\frac{d}{dx} \ln x = \frac{1}{x}, \quad x > 0
\]

If \( u \) is a differentiable function of \( x \) whose values are positive, so that \( \ln u \) is defined, then applying the Chain Rule to the function \( y = \ln u \) gives

\[
\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}, \quad u > 0
\]

(2)

- **Laws of Logarithms.** For any numbers \( a > 0 \) and \( x > 0 \),

1. \( \ln ax = \ln a + \ln x \)
2. \( \ln \frac{a}{x} = \ln a - \ln x \)
3. \( \ln x^n = n \ln x \).

- If \( u \) is a differentiable function that is never zero,

\[
\int \frac{1}{u} \, du = \ln |u| + C.
\]

(28, 30)

- **The Integrals of \( \tan x \) and \( \cot x \).**

\[
\int \tan u \, du = - \ln |\cos u| + C = \ln |\sec u| + C
\]
\[
\int \cot u \, du = \ln |\sin u| + C = - \ln |\csc x| + C
\]

(40)

- **Logarithmic Differentiation.** The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating.

(44)

- **Derivative of \( \log_a u \).** By converting \( \log_a u \) to a natural logarithm, we obtain

\[
\frac{d}{dx} (\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \frac{du}{dx}
\]

(62)

- **Integrals Involving \( \log_a x \).** To evaluate integrals involving base \( a \) logarithms, we convert them to natural logarithms.

(56)
6.2 Exponential Functions

- **Theorem:** *The Derivative Rule for Inverses.* If \( f \) is differentiable at every point of an interval \( I \) and \( df/dx \) is never zero on \( I \), then \( f^{-1} \) is differentiable at every point of the interval \( f(I) \).

  The value of \( df/dx \) at any particular point \( f(a) \) is the reciprocal of the value of \( df/dx \) at \( a \):

\[
\left( \frac{df^{-1}}{dx} \right)_{x=f(a)} = \frac{1}{\left( \frac{df}{dx} \right)_{x=a}}.
\]

In short,

\[
(f^{-1})' = \frac{1}{f'}.
\]

- The function \( \ln x \), being an increasing function of \( x \) with domain \((0, \infty)\) and range \((-\infty, \infty)\), has an inverse \( \ln^{-1} x \) with domain \((-\infty, \infty)\) and range \((0, \infty)\). The graph of \( \ln^{-1} x \) is the graph of \( \ln x \) reflected across the line \( y = x \). As you can see (Figure 6.7),

\[
\lim_{x \to -\infty} \ln^{-1} x = \infty \quad \text{and} \quad \lim_{x \to \infty} \ln^{-1} x = 0.
\]

The number \( e \) is defined by

\[
e = \ln^{-1} 1.
\]

Since \( e \) is positive, \( e^x \) for any rational number \( x \) is positive too. Thus we can take the logarithm and obtain

\[
\ln e^x = x \ln e = x.
\]

Since \( \ln x \) is one-to-one and \( \ln (\ln^{-1} x) = x \), the above equation tells us that \( e^x = \ln^{-1} x \) for \( x \) rational. Since the function \( \ln^{-1} x \) is defined for all \( x \), so we can use it to assign a value to \( e^x \) at every point where \( e^x \) had no previous value. Thus for every real number \( x \),

\[
e^x = \ln^{-1} x.
\]

- **Inverse Equations for** \( e^x \) **and** \( \ln x \).

\[
e^{\ln x} = x \quad \text{(all } x > 0)\]

\[
\ln (e^x) = x \quad \text{(all } x)\]

- The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero. Starting with \( y = e^x \) and performing Logarithmic Differentiation, we get

\[
\frac{d}{dx} e^x = e^x.
\]

Furthermore, if \( u \) is any differentiable function of \( x \), then by the Chain Rule,

\[
\frac{d}{dx} e^u = e^u \frac{du}{dx}.
\]

(2,12)

- The integral of \( e^u \) is

\[
\int e^u du = e^u + C.
\]

(28,60)
• We have defined $e$ as the number for which $ln\ e = 1$. The number $e$ can be calculated as a limit,
\[ \lim_{x \to 0} (1 + x)^{1/x} = e.\]
This can be proved by defining $f(x) = \ln x$, the fact that $f'(1) = 1$, and the definition of derivative. Using a calculator, we get $e \approx 2.71828$.

• Since $a = e^{\ln a}$ for any positive number $a$, we have for any number $x$,
\[ a^x = e^{x \ln a}.\]

• If $a > 0$ and $u$ is a differentiable function of $x$, then $a^u$ is a differentiable function of $x$ and
\[ \frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}. \tag{34,38} \]

• Dividing both sides of the above equation gives
\[ a^u \frac{du}{dx} = \frac{1}{\ln a} \frac{d}{dx} (a^u). \]
Integrating with respect to $x$ then gives
\[ \int a^u du = \frac{a^u}{\ln a} + C. \tag{52} \]