Section 4.1 Exponential Functions

- For \( a > 0 \), the \textbf{exponential function with base} \( a \) is defined by

\[ f(x) = a^x. \]

For \( a \neq 1 \), the domain of \( f \) is \(( -\infty, \infty )\), the range of \( f \) is \(( 0, \infty )\). The graph of \( f \) approaches \( y = 0 \) as \( x \) decreases through negative values if \( a > 1 \), and it approaches \( y = 0 \) as \( x \) increases indefinitely if \( 0 < a < 1 \).

\( \text{eg. } f(x) = 2^x, \ g(x) = \left( \frac{1}{2} \right)^x \) \( 40,26,30 \)

- The number \( e \) is defined as the value that \( \left( 1 + \frac{1}{n} \right)^n \) approaches as \( n \) becomes large. This number is irrational and is approximately equal to 2.72. More accurately, \( e \approx 2.71828 \ldots \)

- The \textbf{natural exponential function} is the exponential function

\[ f(x) = e^x \]

with base \( e \). It is often referred to as the \textbf{exponential function}. \( 48 \)

- If an amount of money \( P \), called the \textbf{principal}, is invested at a simple interest rate \( i \), then after one time period the interest is \( Pi \), and the amount \( A \) of money is

\[ A = P + Pi = P(1 + i). \]

If the interest is reinvested, then after \( k \) periods the amount is

\[ A = P(1 + i)^k. \]

Notice that this is an exponential function with base \( 1 + i \). If the annual interest rate is \( r \) and if interest is compounded \( n \) times per year, then each time period the interest rate is \( i = r/n \), and there are \( nt \) time periods in \( t \) years. So \textbf{compound interest} is calculated by the formula

\[ A(t) = P \left( 1 + \frac{r}{n} \right)^{nt}. \]

- If we let \( m = n/r \), then

\[ A(t) = P \left( 1 + \frac{1}{m} \right)^{mt}. \]

Recall that as \( m \) becomes large, the quantity \( \left( 1 + \frac{1}{m} \right)^m \) approaches the number \( e \). Thus \textbf{continuously compounded interest} is calculated by the formula

\[ A(t) = Pe^{rt}. \] \( 36 \)

Section 4.2 Logarithmic Functions

- Let \( a \) be a positive number with \( a \neq 1 \). The \textbf{logarithmic function with base} \( a \), denoted by \( \log_a \), is defined by

\[ \log_a x = y \iff a^y = x. \]

The first equation is in \textbf{logarithmic form} and the second equivalent equation is in \textbf{exponential form}. In words, this says that \( \log_a x \) is the exponent to which the base \( a \) must be raised to give \( x \).

\( 4,6,8,16,18,20,22,24,30 \)

- Basic Logarithmic Properties Involving One

1. \( \log_a 1 = 0 \)
2. \( \log_a a = 1 \) \( 32,34 \)

- Inverse Properties of Logarithms
1. \( \log_a a^x = x \)
2. \( a^{\log_a x} = x \)

- **Graphs of Logarithmic Functions.** Since the logarithmic function is the inverse of the exponential function, the logarithmic function reverses the coordinates of the exponential function. Therefore the graph of the logarithmic function is a reflection of the graph of the exponential function about the line \( y = x \).

- In Section 4.1 we learned that the domain of an exponential function is \( (-\infty, \infty) \) and its range is \( (0, \infty) \). Because the logarithmic function reverses the domain and the range of the exponential function, the **domain of a logarithmic function is the set of all positive real numbers**.

- The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:
  \[ \log x = \log_{10} x \]

- The logarithm with base \( e \) is called the **natural logarithm** and is denoted by \( \ln \):
  \[ \ln x = \log_e x \]

**Section 4.3 Properties of Logarithms**

- Since logarithms are exponents, the Laws of Exponents give rise to the Laws of Logarithms. Let \( a \) be a positive number, with \( a \neq 1 \). Let \( A > 0 \), \( B > 0 \), and \( C \) be any real numbers. Then

  1. \( \log_a (AB) = \log_a A + \log_a B \)  
     (The Product Rule) 
     [Proof: Let \( u = \log_a A \) and \( v = \log_a B \). Then \( a^u a^v = AB \). So \( \log_a (AB) = \log_a a^{u+v} = \log_a A + \log_a B \).] [2,4]

  2. \( \log_a \left( \frac{A}{B} \right) = \log_a A - \log_a B \)  
     (The Quotient Rule)  
     [8,14]

  3. \( \log_a (A^C) = C \log_a A \)  
     (The Power Rule) 
     [Proof: Let \( u = \log_a A \). Then \( a^u = A \Rightarrow A^C = (a^u)^C \Rightarrow \log_a (A^C) = C \log_a A \).] [16,20,22,28,36,40,44,52]

  - The Change-of-Base Property
    \[ \log_b x = \frac{\log_a x}{\log_a b} \]
    [54]

**Section 4.4 Exponential and Logarithmic Equations**

- An **exponential equation** is one in which the variable occurs in the exponent. [6,10,14,20]

- A **logarithmic equation** is one in which a logarithm of the variable occurs. [26,32,40,46]

**Section 4.5 Applications of Exponential Functions**

- **Compound Interest** [2,6,8,12,14,18,20,22,24]

- **Population Growth** [25,28,30]

- **Radioactive Decay** [40,42,43,44]

- **Newton’s Law of Cooling** states that the temperature of an object placed in a surrounding medium of constant temperature is given by the formula:
  \[ T = S + (T_0 - S) e^{kt} \]

  where

  - \( T \) = the temperature of the object at time \( t \)
  - \( S \) = the temperature of the surrounding medium
  - \( T_0 \) = the initial temperature of the object
  - \( t \) = the time since the object was put in the medium

  and \( k \) is a constant which depends on the specific heats of the object and the medium. [56,57]