Math 1314 Class Notes

Section 3.1 Quadratic Functions

- The graph of any quadratic function is a **parabola**; it can be obtained from the graph of \( y = x^2 \) by transformations.

- A quadratic function \( f(x) = ax^2 + bx + c \) can be expressed in the **standard form**

  \[ f(x) = a(x-h)^2 + k, \quad a \neq 0 \]

  by completing the square. The graph of \( f \) is a parabola with **vertex** \((h,k)\). The parabola is symmetric to the line \( x = h \). The parabola opens upward if \( a > 0 \) or downward if \( a < 0 \).

  - If \( a > 0 \), then the **minimum value** of \( f \) occurs at \( x = h \) and this value is \( f(h) = k \).
  - If \( a < 0 \), then the **maximum value** of \( f \) occurs at \( x = h \) and this value is \( f(h) = k \). [26]

- The maximum or minimum value of a quadratic function \( f(x) = ax^2 + bx + c \) occurs at

  \[ x = -\frac{b}{2a} \]

  - If \( a > 0 \), then the **minimum value** is \( f\left(-\frac{b}{2a}\right) \).
  - If \( a < 0 \), then the **maximum value** is \( f\left(-\frac{b}{2a}\right) \). [32,42]

Section 3.2 Polynomial Functions and Their Graphs

- A polynomial of degree \( n \) is a function of the form

  \[ P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

  where \( a_n \neq 0 \). The numbers \( a_0, a_1, a_2, \ldots, a_n \) are called the **coefficients** of the polynomial. The number \( a_0 \) is the **constant coefficient** or **constant term**. The number \( a_n \) is the **leading coefficient**, and the term \( a_n x^n \) is the **leading term**. If a polynomial consists of just a single term, then it is called a **monomial**.

- The graph of a polynomial function is always a smooth and continuous curve.

- For any polynomial, the **end behavior** is determined by the term that contains the highest power of \( x \), because when \( x \) is large, the other terms are relatively insignificant in size. [22,52]

- If \( P \) is a polynomial and if \( c \) is a number such that \( P(c) = 0 \), then we say that \( c \) is a **zero** of \( P \). The following are equivalent ways of saying the same thing.

  1. \( c \) is a zero of \( P \)
  2. \( x = c \) is a root of the equation \( P(x) = 0 \)
  3. \( x - c \) is a factor of \( P(x) \)
  4. the graph of \( y = P(x) \) has an \( x \)-intercept at \( x = c \)

- If the factor \( x - c \) appears \( k \) times in the complete factorization of \( P(x) \), then we say that \( c \) is a zero of **multiplicity** \( k \). The polynomial \( P \) has the same number of zeros as its degree, provided we count multiplicities.

- Suppose \( c \) is a zero of \( P \) and the corresponding factor \((x - c)^m \) occurs in the factorization of \( P \). Then the graph **crosses** the \( x \)-axis at \( c \) if \( m \) is odd and **touches** the \( x \)-axis and turns around at \( c \) if \( m \) is even. Regardless of whether a zero is even or odd, graphs tend to flatten out at zeros with multiplicity greater than one. [34,28]

- If \( f \) is a polynomial of degree \( n \), then the graph of \( f \) has at most \( n - 1 \) **turning points**. [42]
Section 3.3 Dividing Polynomials; Remainder and Factor Theorems

- In the last section, we studied polynomial functions graphically. In this section we begin to study polynomial algebraically. This requires factoring polynomials, and to factor, we need to know how to divide polynomials.

  - Division Algorithm. If \( P(x) \) and \( D(x) \) are polynomials, with \( D(x) \neq 0 \), then there exist unique polynomials \( Q(x) \) and \( R(x) \) such that
    \[
    P(x) = D(x) \cdot Q(x) + R(x)
    \]
    where \( R(x) \) is either 0 or of degree less than the degree of \( D(x) \). The polynomials \( P(x) \) and \( D(x) \) are called the dividend and divisor, respectively, \( Q(x) \) is the quotient, and \( R(x) \) is the remainder. If \( R(x) = 0 \), we say that \( D(x) \) divides evenly into \( P(x) \) and that \( D(x) \) and \( Q(x) \) are factors of \( P(x) \).

- Synthetic division is a quick method of dividing polynomials; it can be used when the divisor is of the form \( x - c \).

- Remainder Theorem. If the polynomial \( P(x) \) is divided by \( x - c \), then the remainder is the value \( P(c) \).

- Factor Theorem. \( c \) is a zero of \( P \) if and only if \( x - c \) is a factor of \( P(x) \). This tells us that finding the zeros of a polynomial is really the same thing as factoring it into linear factors.

Section 3.4 Zeros of Polynomial Functions

- In this section we study some algebraic methods that help us find the real zeros of a polynomial, and thereby factor the polynomial.

  - Rational Zeros Theorem. If the polynomial \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) has integer coefficients, then every rational zero of \( P \) is of the form \( \frac{p}{q} \) where \( p \) is a factor of the constant coefficient \( a_0 \) and \( q \) is a factor of the leading coefficient \( a_n \).

  - Conjugate Zeros Theorem. If the polynomial \( P \) has real coefficients, and if the complex number \( a + bi \) is a zero of \( P \), then its complex conjugate \( a - bi \) is also a zero of \( P \).

  - (Optional) Descarte’s Rule of Signs. Let \( P \) be a polynomial with real coefficients.
    1. The number of positive real zeros of \( P(x) \) is either equal to the number of variations in sign in \( P(x) \) or is less than that by an even whole number.
    2. The number of negative real zeros of \( P(x) \) is either equal to the number of variations in sign in \( P(-x) \) or is less than that by an even whole number.

Section 3.7 Modeling Using Variation

- Direct Variation. If the quantities \( x \) and \( y \) are related by an equation
  \[
  y = kx
  \]
  for some constant \( k \neq 0 \), we say that \( y \) varies directly as \( x \). The constant \( k \) is called the constant of variation.

- Inverse Variation. If the quantities \( x \) and \( y \) are related by an equation
  \[
  y = \frac{k}{x}
  \]
  for some constant \( k \neq 0 \), we say that \( y \) varies inversely as \( x \).

- Combined Variation. If \( z = \frac{k x}{y} \), we say that \( z \) varies directly as \( x \) and varies inversely as \( y \).

- Joint Variation. If the quantities \( x, y, \) and \( z \) are related by an equation
  \[
  z = kxy
  \]
  for some constant \( k \neq 0 \), we say that \( z \) varies jointly as \( x \) and \( y \).