Math 1314 Class Notes

Section 2.1 Lines and Slope

- The slope \( m \) of a nonvertical line that passes through the points \( A(x_1, y_1) \) and \( B(x_2, y_2) \) is
  \[
  m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.
  \]
  The slope of a vertical line is not defined. [2,8]

- Point-Slope form of the equation of a line: An equation of the line that passes through the point \((x_1, y_1)\) and has slope \(m\) is
  \[
  y - y_1 = m(x - x_1)
  \]
  (derived from definition of slope of a nonvertical line) [12,26]

- Slope-Intercept form of the equation of a line: An equation of the line that has slope \(m\) and \(y\)-intercept \(b\) is
  \[
  y = mx + b
  \]
  (special case of point-slope form) [44]

- Horizontal and Vertical Lines:
  - An equation of the horizontal line through \((a, b)\) is \(y = b\). (special case of slope-intercept form)
  - An equation of the vertical line through \((a, b)\) is \(x = a\). [48,50]

- General Form of the Equation of a Line: The graph of every linear equation
  \[
  Ax + By + C = 0 \quad (A, B \text{ not both zero})
  \]
  is a line. Conversely, every line is the graph of a linear equation. [54]

- Applications [66]

Section P.8 Graphs

- The graph of an equation in \(x\) and \(y\) is the set of all points \((x, y)\) in the coordinate plane that satisfy the equation. [12]

- An \(x\)-intercept of a graph is an \(x\)-coordinate of the point where the graph intersects the \(x\)-axis. The \(y\)-coordinate corresponding to a graph’s \(x\)-intercept is always zero. A \(y\)-intercept of a graph is a \(y\)-coordinate of the point where the graph intersects the \(y\)-axis. The \(x\)-coordinate corresponding to a graph’s \(y\)-intercept is always zero.

- The Distance Formula. The distance \(d\) between the points \((x_1, y_1)\) and \((x_2, y_2)\) in the rectangular coordinate system is
  \[
  d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
  \]

- The Midpoint Formula. Consider a line segment whose endpoints are \((x_1, y_1)\) and \((x_2, y_2)\). The coordinates of the segment’s midpoint are
  \[
  \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).
  \]
  To find the midpoint, take the average of the two \(x\)-coordinates and of the two \(y\)-coordinates. [34,40,42]
Section 2.2 Parallel and Perpendicular Lines and Circles

- Two nonvertical lines are parallel if and only if they have the same slope. \( [22] \)

- Two lines with slopes \( m_1 \) and \( m_2 \) are perpendicular if and only if \( m_1 m_2 = -1 \). Also, a horizontal line (slope 0) is perpendicular to a vertical line (undefined slope). \( [24] \)

- A circle is the set of all points in a plane that are equidistant from a fixed point called the center. The fixed distance from the circle’s center to any point on the circle is called the radius.

- The standard form for the equation of the circle with center \((h, k)\) and radius \( r \) is
  \[
  (x - h)^2 + (y - k)^2 = r^2.
  \]
  \([30,32,44]\)

- The general form of the equation of a circle is
  \[
  x^2 + y^2 + Dx + Ey + F = 0.
  \]
  \([52]\)

Section 2.3 Introduction to Function

- A set of ordered pairs is a relation. When a set of ordered pairs is such that no two different pairs share a common first coordinate, we have a function. The domain of a function is the set of all first coordinates and the range is the set of all second coordinates. \([2,4]\)

- A function is a rule that describes how one quantity depends on another. We use the term function to describe this dependence of one quantity on another. For example:
  - Height is a function of age
  - Cost of mailing a package is a function of weight

The U.S. Post Office uses a simple rule to determine the cost of mailing a package based on its weight. But it’s not so easy to describe the rule that relates height to age.

- Other example: The number \( N \) of bacteria in a culture depends on the time \( t \). The rule that connects \( N \) and \( t \) in this case is given by the formula \( N = 50 \cdot 2^t \). We can visualize a function by sketching its graph. The height of the graph above the horizontal axis represents the number of bacteria at a given time \( t \)—the higher the graph, the more bacteria.

- The symbol that represents an arbitrary number in the domain of a function is called an independent variable. The symbol that represents a number in the range of a function is called a dependent variable. For instance, in the bacteria example, \( t \) is the independent variable and \( N \) is the dependent variable. \([12,14]\)

- If the letter \( f \) represents a function, then the notation \( f (x) \) means “apply the rule \( f \) to the number \( x \)”, and it is read “\( f \) of \( x \)”. \([24,46]\)

- A function \( f \) is a rule that assigns to each element \( x \) in a set \( A \) exactly one element, called \( f (x) \), in a set \( B \). The set \( A \) is called the domain of the function. The range of \( f \) is the set of all possible values of \( f (x) \) as \( x \) varies throughout the domain.
The process of finding a function to describe a real-world phenomenon is called **modeling**. (For instance, a biologist observes that the number of bacteria in a culture increases with time. To predict the size of the bacteria culture in the future, the biologist tries to find the rule or function that relates the number of bacteria to the time.)

**Finding a Function’s Domain.** If a function \( f \) does not model data or verbal conditions, its domain is the largest set of real numbers for which the value of \( f(x) \) is a real number. Exclude from a function’s domain real numbers that cause division by zero and real numbers that result in an even root of a negative number.

**Section 2.4 Graphs of Functions**
- The graph of \( f \) is the set of all points \((x,y)\) such that \( y = f(x) \); that is, the graph of \( f \) is the graph of the equation \( y = f(x) \). (eg. \( f(x) = x^2 \), \( g(x) = x^3 \), \( h(x) = \sqrt{x} \))

**The Vertical Line Test.** A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

**Increasing, Decreasing, and Constant Functions.**
- \( f \) is **increasing** on an interval \( I \) if \( f(x_1) < f(x_2) \) whenever \( x_1 < x_2 \) in \( I \).
- \( f \) is **decreasing** on an interval \( I \) if \( f(x_1) > f(x_2) \) whenever \( x_1 < x_2 \) in \( I \).
- \( f \) is **constant** on an interval \( I \) if \( f(x_1) = f(x_2) \) whenever \( x_1 < x_2 \) in \( I \).

**The function** \( f \) **is an even function** if

\[
f(-x) = f(x) \quad \text{for all } x \text{ in the domain of } f.
\]

**The function** \( f \) **is an odd function** if

\[
f(-x) = -f(x) \quad \text{for all } x \text{ in the domain of } f.
\]

- The graph of an even function in which \( f(-x) = f(x) \) is symmetric with respect to the \( y \)-axis. The graph of an odd function in which \( f(-x) = -f(x) \) is symmetric with respect to the origin.

**Section 2.5 Transformations and Combinations of Functions**
- **Vertical Shifting.** Adding a constant to a function shifts its graph vertically: upward if the constant is positive and downward if it is negative.

- **Horizontal Shifting.** Suppose we know the graph of \( y = f(x) \). The value of \( f(x-c) \) at \( x \) is the same as the value of \( f(x) \) at \( x-c \). Since \( x-c \) is \( c \) units to the left of \( x \), it follows that the graph of \( y = f(x-c) \) is just the graph of \( y = f(x) \) shifted to the right \( c \) units. Similar reasoning shows that the graph of \( y = f(x+c) \) is the graph of \( y = f(x) \) shifted to the left \( c \) units.

- **Reflecting Graphs.** The graph of \( y = -f(x) \) is the reflection of the graph of \( y = f(x) \) in the \( x \)-axis. On the other hand, the value of \( y = f(-x) \) at \( x \) is the same as the value of \( y = f(x) \) at \(-x\) and so the graph of \( y = f(-x) \) is the reflection of the graph of \( y = f(x) \) in the \( y \)-axis.
- **Vertical Stretching and Shrinking.** Multiplying a function by \( a \) has the effect of vertically stretching or shrinking the graph by a factor of \( a \). [20,42]

- **Combinations of Functions.** Let \( f \) and \( g \) be functions with domains \( A \) and \( B \). Then the functions \( f + g \), \( f - g \), \( fg \), and \( f/g \) are defined as followings.

\[
(f + g)(x) = f(x) + g(x) \quad \text{Domain } A \cap B \\
(f - g)(x) = f(x) - g(x) \quad \text{Domain } A \cap B \\
(fg)(x) = f(x)g(x) \quad \text{Domain } A \cap B \\
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{Domain } \{x \in A \cap B | g(x) \neq 0\}
\]

Section 2.6 Composite and Inverse Functions

- **Composition of Functions.** Given two functions \( f \) and \( g \), the composite function \( f \circ g \) is defined by

\[
(f \circ g)(x) = f(g(x))
\]

The domain of \( f \circ g \) is the set of all \( x \) such that

1. \( x \) is in the domain of \( g \) and
2. \( g(x) \) is in the domain of \( f \). [6]

- **Definition.** Let \( f \) and \( g \) be two functions such that

\[
f(g(x)) = x \quad \text{for every } x \text{ in the domain of } g
\]

and

\[
g(f(x)) = x \quad \text{for every } x \text{ in the domain of } f.
\]

The function \( g \) is the inverse of the function \( f \), and is denoted by \( f^{-1} \) (read “\( f \)-inverse”). Thus, \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \). The domain of \( f \) is equal to the range of \( f^{-1} \), and vice versa. [16,18]

- **Finding the Inverse of a Function.** The equation for the inverse of a function \( f \) can be found as follows.

1. Replace \( f(x) \) by \( y \) in the equation for \( f(x) \).
2. Interchange \( x \) and \( y \).
3. Solve for \( y \). If this equation does not define \( y \) as a function of \( x \), the function \( f \) does not have an inverse function and this procedure ends. If this equation does define \( y \) as a function of \( x \), the function \( f \) has an inverse function.
4. If \( f \) has an inverse function, replace \( y \) in step 3 by \( f^{-1}(x) \). We can verify our result by showing that \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \). [26,32]

- **The Horizontal Line Test For Inverse Functions.** A function \( f \) has an inverse that is a function, \( f^{-1} \), if there is no horizontal line that intersects the graph of the function \( f \) at more than one point. [46,48]

- A **one-to-one function** is a function in which no two different ordered pairs have the same second component. Only one-to-one functions have inverse functions. A function passes the horizontal line test if and only if it is a one-to-one function.

- The graph of \( f^{-1} \) is a reflection of the graph of \( f \) about the line \( y = x \). [52]