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SEMI-HOMOGENEOUS BASES IN ORLICZ SEQUENCE SPACES

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Abstract. A Schauder basis (x_n) of a Banach space X is said to be lower [upper] semi-homogeneous if every normalized block basic sequence of (x_n) dominates [is dominated by] the basis. We show that the unit vector basis of a separable Orlicz sequence space ℓ_M, with $M(1) = 1$ is lower [upper] semi-homogeneous if and only if M fulfils some kind of super[sub]-multiplicative inequality on the interval$[0, 1]$. We present both explicit examples and a general method of constructing such functions not equivalent to a power function. We also show that if the unit vector basis of a reflexive Orlicz sequence space ℓ_M is semi-homogeneous, then the set of p's for which ℓ_p is isomorphic to a subspace of ℓ_M is a singleton.

1. Introduction

Let X be a Banach space with a normalized Schauder basis (x_n). The basis is said to be perfectly homogeneous if it is equivalent to all of its normalized block bases. The classical result of Zippin asserts that (x_n) is perfectly homogeneous if and only if it is equivalent to the unit vector basis of c_0 or ℓ_p for some $1 \leq p < \infty$ ([Z]; [LT], Theorem 2.a.9). In 1975 Casazza and Lin defined (x_n) to be *lower* (resp., *upper*) semi-homogeneous if every normalized block basic sequence of the basis dominates (resp., is dominated by) (x_n), and they characterized Lorentz sequence spaces possessing a lower semi-homogeneous unit vector basis ([CL]).

In this paper, we give characterizations of separable Orlicz sequence spaces which admit a semi-homogeneous basis, and we provide examples of such bases which are not perfectly homogeneous. In Theorem 3.1 we show that if ℓ_M is a separable Orlicz sequence space then its unit vector basis is upper [resp., lower] semi-homogeneous iff M fulfils some kind of sub-multiplicative [resp., super-multiplicative] inequality on the interval [0,1]. We also show that refelxive Orlicz sequence spaces with semi-homogeneous bases are 'tight' in the sense that the interval of p's for which ℓ_p embeds into such a space ℓ_M reduces to a singleton (Corollary 4.5).

Our results are motivated not only by the above-cited paper by Casazza and Lin, but also by the role of Orlicz sequence spaces in the illustration of the theory of spreading models as well (see the recent papers [DOS, S]).

2. Preliminaries

We follow the terminology of the monographs [LT] and [Sin]. By c_{00} we denote the linear space of finite scalar sequences. Let (x_n) and (y_n) be normalized bases of Banach spaces X and Y, respectively. We say that (y_n) dominates (or C-dominates)

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(x_n) if there exists a constant $C \geq 1$ such that for all $(a_n) \in c_{00}$,

$$\|\sum_n a_n x_n\| \leq C \|\sum_n a_n y_n\|.$$

The bases (x_n) and (y_n) are equivalent if (x_n) both dominates and is dominated by (y_n). A sequence (u_n) is a normalized block basis of (x_n) if it is of the form $u_n = \sum_{j=p(n)+1}^{p(n+1)} a_j x_j$, where $(p(n))$ is a strictly increasing sequence of positive integers, and (a_j) is a sequence of scalars such that $\|u_n\| = 1$. The basis is *subsymmetric* if it is unconditional and equivalent to every of its subbasis (x_{k_n}), and it is called *symmetric* if it is equivalent to $(x_{\pi(n)})$ for every permutation π of $\mathbb{N}$.

The following lemma (included implicitly in the proof of ([Sin], Proposition 24.2); cf. [CL], p. 139) asserts that in the definition of subsymmetric semi-homogeneous bases the domination constants can be chosen uniformly over all block bases.

Lemma 2.1. *Let (x_n) be a subsymmetric normalized basis of a Banach space X.*

(*a*) *If (x_n) is lower semi-homogeneous, then there exists a constant $C \geq 1$ such that every normalized block basis of (x_n) C-dominates (x_n).*

(*b*) *If (x_n) is upper semi-homogeneous, then there exists a constant $C \geq 1$ such that (x_n) C-dominates every normalized block basis of (x_n).*

Let $M : \mathbf{R}^+ \to \mathbf{R}^+$ be an Orlicz function, i.e., non-decreasing, convex, with $M(0) = 0$. Without loss of generality we may assume that $M(1) = 1$. We consider also convex functions defined on finite intervals $[0, t_0]$ which can be extended to Orlicz functions, linearly on $[t_0, \infty)$.

Let ω denote the space of all scalar sequences. We define the function $\varrho_M : \omega \to [0, \infty]$ by the formula $\varrho_M(x) = \sum_{n=1}^{\infty} M(|t_n|)$, where $x = (t_n) \in \omega$. The Orlicz sequence space ℓ_M is defined as the linear set

$$\ell_M := \{x \in \omega : \varrho_M(x/\lambda) < \infty \text{ for some } \lambda > 0\},$$

and equipped with the norm $\|x\|_M := \inf\{\lambda > 0 : \varrho_M(x/\lambda) \leq 1\}$ it becomes a Banach space. By h_M we denote the closed subspace of ℓ_M defined as

$$h_M := \{x \in \omega : \varrho_M(x/\lambda) < \infty \text{ for all } \lambda > 0\},$$

where the unit vectors (e_i) form a symmetric Schauder basis of h_M ([LT], Proposition 4.a.2). Notice that if M is degenerate (i.e., M vanishes on some interval $[0, t_0]$ with $t_0 > 0$), then h_M is isomorphic to c_0 and ℓ_M is isomorphic to ℓ_∞. If M fulfils the so-called Δ_2-condition at 0, i.e., there is $t_0 > 0$ with $M(2t) \leq K \cdot M(t)$ for some $K > 0$ and all $t \in [0, t_0]$, then the spaces ℓ_M and h_M coincide (equivalently, ℓ_M *is separable* ([LT], Proposition 4.a.40), and in this case M is necessarily non-degenerate).

Let M, N be two Orlicz functions. We say that M *dominates* N (at 0) if there exist positive numbers K, k, t_0 such that $M(t) \geq KN(kt)$ for all $t \in [0, t_0]$. The functions are *equivalent at* 0 if M dominates N and N dominates M; in this case the spaces ℓ_M and ℓ_N (as well as h_M and h_N) are isomorphic (more exactly, $\ell_M = \ell_N$ and $h_M = h_N$ as sets; see [LT], p. 139).

The following simple observation shows us that the above-defined properties of Orlicz functions (i.e., the Δ_2 condition and equivalence at 0) can be considered on arbitrary intervals $[0, a]$, $a > 0$ (cf. [LT], Remark on p. 141). Since the function $t \mapsto M(2t)/M(t)$ is continuous on every interval $[t_0, a]$, $a > t_0$, it attains its maximum, say $K_1(a)$. Then, if M satisfies condition Δ_2 on $[0, t_0]$ with a constant K, the condition holds also on $[0, a]$ with the constant $K(a) = \max\{K, K_1(a)\}$. A similar remark relates to the function $t \to N(t)/M(t)$ on $[t_0, a]$ whenever M dominates N. For the reference purposes we list these properies and their immediate consequences in the lemma below.

Lemma 2.2. *Let M, N be two Orlicz functions.*

(a) *If M fulfils condition Δ_2 at 0 then, for every $a > 0$, there is a constant $K(a) > 0$ with $M(2t) \leq K(a)M(t)$ for all $t \in [0, a]$. In particular, for every $D > 0$ there is a constant K_D such that $M(Dt) \leq K_D M(t)$ for all $t \in [0, 1]$.*

(b) *If M and N are equivalent at 0 then, for every $a > 0$, there exist constants $K = K(a), k = k(a)$ such that $K^{-1}M(k^{-1}t) \leq N(t) \leq KM(kt)$ for all $t \in [0, a]$. If, moreover, M (hence N too) fulfils condition Δ_2 at 0, then (by the above item (a)) in the latter inequality we can take $k = 1$.*

3. Semi-homogeneous bases in h_M

Our main result gives a characterization of Orlicz sequence spaces h_M with a semi-homogeneous basis in terms of the multiplicative properties of M. Here we consider only non-degenerate Orlicz functions since the space h_M is isomorphic to c_0 otherwise, whose natural Schauder basis is evidently homogeneous and the problem we deal with has (in this case) a trivial solution.

Theorem 3.1. *Let M be a non-degenerate Orlicz function with $M(1) = 1$, and let (e_i) denote the unit vector basis of the space h_M. The basis is*

(a) *lower semi-homogeneous if and only if there exists a number $D \geq 1$ such that $M(Dst) \geq M(s)M(t)$ for all $s, t \in [0, 1]$.*

(b) *upper semi-homogeneous if and only if there exists a number $D \geq 1$ such that $M(st) \leq M(Ds)M(t)$ for all $s, t \in [0, 1]$.*

If, moreover, M satisfies condition Δ_2 at 0 (and then $h_M = \ell_M$) then the above conditions (a) and (b) are equivalent to the respective conditions below:

(a') *$CM(st) \geq M(s)M(t)$, for all $s, t \in [0, 1]$ and some $C \geq 1$,*

(b') *$M(st) \leq CM(s)M(t)$, for s, t, C as above.*

A function $f : [0, 1] \to \mathbf{R}^+$ is said to be super-multiplicative if $f(st) \geq f(s)f(t)$ for all $s, t \in [0, 1]$; if the latter inequality holds in reverse, then f is said to be sub-multiplicative. Hence, inequalities in the above conditions (a) and (b), resp., imply that the function M is super- and sub-multiplicative, resp., on $[0, 1]$ whenever $D = 1$. Similarly, conditions (a') and (b'), resp., mean that the function M/C is super- and sub-multiplicative, resp., on $[0, 1]$ for the *given* constant C.

Proof. We shall prove only parts (a) and (a') and outline the proof of part (b) (which can be obtained in a similar way).

"'Only if"'. For every $k \in \mathbb{N}$ we put $c_k = \| \sum_{j=1}^{k} e_j \|$, and

$$u_n^{(k)} = (1/c_k) \sum_{m=k\cdot(n-1)+1}^{kn} e_m, \ n = 1, 2, \dots .$$

Since the basis is symmetric and its symmetric constant equals 1, the sequence $(u_n^{(k)})_{n=1}^{\infty}$ is a normalized block basic sequence in h_M. From the definition of the norm in ℓ_M and continuity of M we obtain that

$$c_k = (M^{-1}(1/k))^{-1}, \ k = 1, 2, \dots . \quad (1)$$

By Lemma 2.1 (a), there is a constant $C_0 \geq 1$ such that $\|\sum_{n=1}^r e_n\| \leq C_0 \cdot \|\sum_{n=1}^r u_n^{(k)}\|$ for all $k, r \geq 1$. From the form of $u_n^{(k)}$ we easily obtain that $\sum_{n=1}^r u_n^{(k)} = (1/c_k)\sum_{n=1}^{kr} e_n$. Hence the latter inequality is equivalent to $c_r \leq C_0 \cdot (c_{kr}/c_k)$, i.e., by (1),

$$M^{-1}(\frac{1}{kr}) \leq C_0 \cdot M^{-1}(1/k)M^{-1}(1/r) \text{ for all } k, r \geq 1 \ . \tag{2}$$

Now let us fix $s, t \in (0, 1]$. Then there exist (unique) $k, r \in \mathbb{N}$ with $\frac{1}{k+1} < s \leq \frac{1}{k}$ and $\frac{1}{r+1} < t \leq \frac{1}{r}$. Since $1/k \leq 2s$ and $1/r \leq 2t$, from (2) and the monotonicity and subadditivity of M^{-1} we obtain

$$\begin{aligned} M^{-1}(st) &\leq C_0 \cdot M^{-1}(1/k)M^{-1}(1/r) \\ &\leq C_0 M^{-1}(2s)M^{-1}(2t) \leq 4C_0 \cdot M^{-1}(s)M^{-1}(t) \\ &= D \cdot M^{-1}(s)M^{-1}(t), \end{aligned}$$

where $D = 4C_0$. (We can take the infimum of all D's fulfilling the above inequality, obtaining $D \geq 1$ and not merely $D \geq 4$).

Now we put $s' = M(s)$ and $t' = M(t)$. From the latter inequality we obtain $M(Dst) = M(D \cdot M^{-1}(s')M^{-1}(t')) \geq s't' = M(s)M(t)$, as desired.

"'If"'. Let (A_n) be a sequence of finite subsets of $\mathbb{N}$ with $\max A_n < \min A_{n+1}$ for all n's, and let (θ_j) be a sequence of arbitrary numbers such that the elements $u_n \in h_M$, $n = 1, 2, \ldots$, of the form $u_n := \sum_{j \in A_n} \theta_j x_j$ are of norm one. Equivalently,

$$\sum_{j \in A_n} M(|\theta_j|) = 1 \text{ for all } n's. \tag{3}$$

Assume that a series $\sum_{n=1}^\infty t_n u_n$ converges in h_M, i.e., (see the definition of h_M)

$$\sum_{n=1}^\infty \sum_{j \in A_n} M(|\theta_j| \cdot |t_n/\lambda|) < \infty \text{ for all } \lambda > 0. \tag{4}$$

We claim that *the numerical series* $\sum_{n=1}^\infty M(t_n/\lambda)$ *converges for every* $\lambda > 0$ *too*, i.e., $\sum_{n=1}^\infty t_n x_n \in h_M$. To this end, fix $\lambda > 0$, put $\mu = \lambda/D$, and observe that inequality (4) holds in paricular for μ:

$$\sum_{n=1}^\infty \sum_{j \in A_n} M(|\theta_j| \cdot |t_n/\mu|) < \infty. \tag{5}$$

Moreover, by (3), we have $|\theta_j| \leq 1$ for all j's and, by (4), there is $n_\lambda \in \mathbb{N}$ with $|t_n/\lambda| \leq 1$ for all $n \geq n_\lambda$. Hence, by (3), (5) and the assumed inequality in (a), we obtain

$$\begin{aligned} \infty > \sum_{n=n_\lambda}^\infty \sum_{j \in A_n} M(|\theta_j| \cdot |t_n/\mu|) &= \sum_{n=n_\lambda}^\infty \sum_{j \in A_n} M(|\theta_j| \cdot D|t_n/\lambda|) \\ &\geq \sum_{n=n_\lambda}^\infty \big(\sum_{j \in A_n} M(|\theta_j|)\big) \cdot M(|t_n/\lambda|) \\ &= \sum_{n=n_\lambda}^\infty M(|t_n/\lambda|). \end{aligned}$$

This proves our claim, and hence we obtain that the block basic sequence (u_n) dominates (e_n). Since (u_n) was arbitrary, we obtain that (x_n) is lower semi-homogeneous indeed.

In the proof of the "'only if"' implication in part (b) of our theorem we apply Lemma 2.1 (b) with the symbols $u_n^{(k)}$ and c_k as above, and we obtain

$$M^{-1}(1/kr) \geq C_0 \cdot M^{-1}(1/k)M^{-1}(1/r) \text{ for all } k, r \geq 1 , \tag{6}$$

where $C_0 \in (0,1]$. Now we fix $s,t \in (0,1]$, with $\frac{1}{k+1} < s \leq \frac{1}{k}$ and $\frac{1}{r+1} < t \leq \frac{1}{r}$ for some $k, r \geq 1$. The concavity of M^{-1} yields $M^{-1}(1/(k+1)) = M^{-1}(\frac{1}{k}\frac{k}{k+1}) \geq M^{-1}(\frac{1}{2k}) \geq \frac{1}{2}M^{-1}(1/k) \geq \frac{1}{2}M^{-1}(s)$; similarly $M^{-1}(1/(r+1)) \geq \frac{1}{2}M^{-1}(t)$. Hence, by (6), we obtain

$$M^{-1}(st) \geq M^{-1}\left(\frac{1}{(k+1)(r+1)}\right) \geq \frac{C_0}{4}M^{-1}(s)M^{-1}(t).$$

The remaining steps in the proof of part (b) are similar to their counterparts in the proof of part (a).

Now assume that M satisfies condition Δ_2 at 0. Then, by Lemma 2.2 (b), from the inequality in part (a) of our theorem we obtain (a'). On the other hand, the proof that (a') implies the lower semi-homogeneity of (e_n) is almost the same as the proof for the case (a). A similar argument shows that conditions (b') and (b) are equivalent too.

□

Let M be an Orlicz function with $M(1) = 1$ and satisfying condition Δ_2 at 0. From Theorem 3.1 it now follows that if M is sub[or super]-multiplicative on $[0,1]$ and not equivalent to a power function, then the unit vector basis (e_i) of ℓ_M is upper [or lower] semi-homogeneous and not perfectly homogeneous. Below we present a class of such functions.

Example 3.2. Let a, b, p be *fixed* real numbers with $a \neq 0$, $b > 0$, and $p > 1$. Let $M_{a,b,p}$ be the function defined on the interval $[0, 1/b)$ by the formulas $M_{a,b,p}(0) = 0$, and

$$M_{a,b,p}(x) = x^p|\log(bx)|^a \quad \text{for } x \neq 0.$$

(The cases $b = 1 = a$ and $a > 0$ were studied in ([L], Example 3.4) and ([LT], Example 4.c.1), respectively.) It easy to check that $M_{a,b,p}(0^+) = 0$, and that for $x > 0$ the first and the second derivatives of $M_{a,b,p}$ have the forms

$$M'_{a,b,p}(x) = M_{a-1,b,p-1}(x)(p|\log(bx)| - a),$$

and

$$M''_{a,b,p}(x) = A(a,b,p,x)\left[\left(p + \frac{a}{\log(bx)}\right)\left(p - 1 + \frac{a-1}{\log(bx)}\right) + \frac{p}{\log(bx)}\right],$$

where $A(a,b,p,x) = (\log(bx))^2 M_{a-2,b,p-2}(x)$. Thus, *the both derivatives are positive on an interval* $[0, t_0]$ *with* $t_0 \in (0, 1/b)$, and hence $M_{a,b,p}$ *extends - linearly on* $[t_0, \infty)$ *- to an Orlicz function, denoted further also by* $M_{a,b,p}$, and it can be easily seen that $M_{a,b,p}$ is not equivalent at 0 to a power function.

For $x \in (0, t_0)$ we have

$$\frac{xM'_{a,b,p}(x)}{M_{a,b,p}(x)} = p - \frac{a}{|\log(bx)|},$$

whence $\lim_{x\to 0} xM'_{a,b,p}(x)/M_{a,b,p}(x) = p$. Thus, by ([LT], Proposition 4.b.2') and the remark following ([LT], Proposition 4.b.10), we obtain: *The Orlicz space* $\ell_{M_{a,b,p}}$ *is reflexive* (and hence $\ell_{M_{a,b,p}} = h_{M_{a,b,p}}$) *and its unit vector basis* (e_i) *is, up to equivalence, a unique symmetric basis.* Now we claim that

- *The basis* (e_i) *is upper semi-homogeneous for* $a > 0$*, and lower semi-homogeneous for* $a < 0$*.*

To simplify our notations, we put $M = M_{a,b,p}$, and let us define the Orlicz function N by the formula $N(x) := M(t_0x)/M(t_0)$, $x \geq 0$. Then N is evidently equivalent to M at 0 with $N(1) = 1$. Consider the function $f : (-\infty, \log t_0] \to \mathbf{R}$ of the form $f(t) = \log M(e^t)$. Hence, $f(t) = pt + a \log|\log b + t|$, and so $f''(t) = -a/(\log b + t)^2$. Thus, f is concave for $a > 0$ and convex for $a < 0$. By Theorem 1 (ii) of [FW] the function N is sub-multiplicative on $[0, 1]$ for $a > 0$ and super-multiplicative for $a < 0$. Hence, by parts (a') and (b') of Theorem 3.1, the unit vector basis of ℓ_M is upper semi-homogeneous for $a > 0$ and lower semi-homogeneous for $a < 0$, as claimed.

4. Multiplicative Orlicz functions and sequences of zeros and ones

In this section we give a general method of constructing examples of sub- and super-multiplicative Orlicz functions using the representations of Orlicz functions by sequences of zeros and ones, introduced by Lindenstrauss and Tzafriri (see [LT], p. 161). This will lead us to another characterization of reflexive Orlicz sequence spaces with semi-homogeneous bases, and a structure result about ℓ_p subspaces of such spaces.

First, we recall this method (see [LT] for more details). Fix $0 < \tau < 1$ and $1 < r < p < \infty$. For every sequence of zeros and ones, $\rho = (\rho(n))_{n=1}^{\infty}$ (i.e. $\rho(n) \in \{0, 1\}$ for all n), let M_ρ be the piecewise linear function defined on $[0, 1]$ satisfying $M_\rho(0) = 0$, $M_\rho(1) = 1$, and

$$M_\rho(\tau^k) = \tau^{rk+(p-r)\sum_{n=1}^{k}\rho(n)}, \qquad k = 1, 2, \ldots.$$

It is known (see [DOS], Lemma 3.3) that M_ρ is convex whenever $p-r$ is sufficiently small, and that every Orlicz function M such that ℓ_M is refexive is equivalent at 0 to some Orlicz function of the form M_ρ ([LT], Proposition 4.c.8).

Using this method one can construct many examples of reflexive Orlicz sequence spaces (not isomorphic to c_0 or ℓ_p) with a semi-homogeneous basis. The example below comes from ([DOS], Lemma 3.5).

Example 4.1. Let a sequence $(n_k) \subset \mathbb{N}$ satisfy $n_1 = 1$ and $n_{k+1} - n_k \uparrow \infty$ (e.g., $n_k = 2^k$, $k = 0, 1, 2, \ldots$), and put

$$\rho_j(i) = \begin{cases} 0 & \text{if } i = n_k \\ 1 & \text{otherwise.} \end{cases}$$

(e.g., $\rho = (0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 0, \ldots)$) Let the constants $1 < r < p < \infty$ and $0 < \tau < 1$ be chosen for the corresponding function $M := M_\rho$ to be convex (i.e., an Orlicz function). Then the unit vector basis of ℓ_M is upper semi-homogeneous.

To see this, by Δ_2-condition and Theorem 3.1, it is sufficient to check that

$$M(st) \leq M(s)M(t) \text{ for all } 0 \leq s = \tau^k, t = \tau^m \leq 1.$$

This is a straightforward verification: indeed, observe that ρ satisfies

$$\sum_{n=1}^{k+m} \rho(n) \geq \sum_{n=1}^{k} \rho(n) + \sum_{n=1}^{m} \rho(n), \text{ for all } k, m \in \mathbb{N}.$$

Thus,

$$\begin{aligned} M(st) &= M(\tau^{k+m}) = \tau^{r(k+m)+(p-r)\sum_{n=1}^{k+m}\rho(n)} \\ &\leq \tau^{rk+rm+(p-r)\left(\sum_{n=1}^{k}\rho(n)+\sum_{n=1}^{m}\rho(n)\right)} \\ &= \tau^{rk+(p-r)\sum_{n=1}^{k}\rho(n)}\ \tau^{rm+(p-r)\sum_{n=1}^{m}\rho(n)} = M(s)M(t). \end{aligned}$$

We note here that ℓ_M contains only two non-equivalent symmetric basic sequences, namely, the unit vector basis of ℓ_M and of ℓ_p. This can be checked by showing that the set $C_{M,1}$ contains, up to equivalence, only M and t^p; see the proof of Lemma 3.5 in [DOS].

By exchanging the 0's and 1's in the definition of ρ we get a lower semi-homogeneous example. In this case the space ℓ_M contains uncountably many mutually non-equivalent symmetric basic sequences. This follows from Theorem 3.3 in [S]. However, the only power function in $C_{M,1}$ is still t^p. That is, in both of these examples the interval of p's for which ℓ_p is isomorphic to a subspace of ℓ_M is a singleton.

From the discussion in the above example, the following is now obvious.

Lemma 4.2. *Every Orlicz function M_ρ where ρ satisfies: There exists a constant K such that*

$$K + \sum_{n=1}^{k+m} \rho(n) \geq \sum_{n=1}^{k} \rho(n) + \sum_{n=1}^{m} \rho(n), \text{ for all } k, m \in \mathbb{N} \tag{7}$$

yields an Orlicz sequence space whose unit vector basis is upper semi-homogeneous. Similarly, those M_ρ's satisfying: There exists a constant K such that

$$\sum_{n=1}^{k+m} \rho(n) \leq \sum_{n=1}^{k} \rho(n) + \sum_{n=1}^{m} \rho(n) + K, \text{ for all } k, m \in \mathbb{N} \tag{8}$$

yield Orlicz sequence spaces whose unit vector bases are lower semi-homogeneous.

In fact, the above conditions characterize reflexive Orlicz sequence spaces with semi-homogeneous bases.

Theorem 4.3. *Let ℓ_M be a reflexive Orlicz sequence space.*

- (i) *The unit vector basis of ℓ_M is upper semi-homogeneous iff there exist $1 < r < p < \infty$, $0 < \tau < 1$, and a sequence ρ of zeros and ones satisfying (7) such that the corresponding function M_ρ is equivalent to M at 0.*
- (ii) *The unit vector basis of ℓ_M is lower semi-homogeneous iff there exist $1 < r < p < \infty$, $0 < \tau < 1$, and a sequence ρ of zeros and ones satisfying (8) such that the corresponding function M_ρ is equivalent to M at 0.*

Proof. We will show only (i) as the proof of (ii) is similar. Suppose ℓ_M is reflexive. Then, by Proposition 4.c.8 in [LT], there exist $1 < r < p < \infty$, $0 < \tau < 1$, and a sequence ρ of zeros and ones such that the corresponding function M_ρ is equivalent to M at 0. Since the unit vector basis of ℓ_M is upper semi-homogeneous, by Theorem 3.1, the function M satisfies $M(st) \leq CM(s)M(t)$ for all $s, t \in [0, 1]$. This and part (b) of Lemma 2.2 imply that $M_\rho(st) \leq CC'M_\rho(s)M_\rho(t)$ for $s, t \in [0, 1]$, where C' is the equivalence constant between M and M_ρ. (From the proof of Proposition 4.c.8 in [LT] one can trace the exact value of C'.)

Let $k, m \in \mathbb{N}$ and $s = \tau^k$, $t = \tau^m$, and let $K \in \mathbb{N}$ such that $CC' \le \tau^{-K(p-r)}$. Then,

$$\begin{aligned} \tau^{r(k+m)+(p-r)\sum_{n=1}^{k+m}\rho(n)} &= M_\rho(\tau^{k+m}) = M_\rho(st) \le CC' M_\rho(s) M_\rho(t) \\ &\le \tau^{-K(p-r)} \tau^{rk+(p-r)\sum_{n=1}^{k}\rho(n)}\ \tau^{rm+(p-r)\sum_{n=1}^{m}\rho(n)} \\ &= \tau^{r(k+m)+(p-r)\left(\sum_{n=1}^{k}\rho(n)+\sum_{n=1}^{m}\rho(n)-K\right)}. \end{aligned}$$

That is,

$$K + \sum_{n=1}^{k+m} \rho(n) \ge \sum_{n=1}^{k} \rho(n) + \sum_{n=1}^{m} \rho(n),$$

as desired. By Lemma 2.2 (b), the converse can be checked similarly as in Example 4.1. □

Our examples so far suggest that the set of p's for which ℓ_p is isomorphic to a subspace of ℓ_M with a semi-homogenous basis is a singleton. Indeed, this is the case for reflexive spaces as we show below. Recall (see Proposition 4.c.4, [LT]) that the set of p's for which ℓ_p embeds into ℓ_{M_ρ} is the interval $[\alpha_{M_\rho}, \beta_{M_\rho}]$ where

$$\alpha_{M_\rho} = r + (p-r) \liminf_k \inf_n \frac{1}{k} \sum_{i=n+1}^{n+k} \rho(i)$$

and

$$\beta_{M_\rho} = r + (p-r) \limsup_k \sup_n \frac{1}{k} \sum_{i=n+1}^{n+k} \rho(i).$$

Theorem 4.4. *Let $M := M_\rho$ be an Orlicz function determined by a sequence ρ and parameters (τ, r, p). If ρ satisfies (8) or (7) then $\alpha_M = \beta_M$.*

Proof. First, note that $\lim_n \frac{P(n)}{n}$, where $P(n) = \sum_{i=1}^{n} \rho(i)$, exists. We reproduce the proof of this known fact for convenience of the reader (cf. [GRS], p. 45). Indeed, suppose that ρ satisfies (8). Set $\alpha = \limsup_{n\to\infty} \frac{P(n)}{n}$. Let $\varepsilon > 0$, and fix $n \in \mathbb{N}$ such that $K/n \le \varepsilon$. For $m \in \mathbb{N}$ write $m = qn + r$, where $0 \le r < n$, and thus

$$P(m) = P(qn+r) \le P((q+1)n) \le (q+1)[P(n)+K].$$

Hence we obtain $\dfrac{P(m)}{m} \le \dfrac{(q+1)[P(n)+K]}{qn}$. Now we make m tend to infinity, so will do q, to obtain $\alpha \le P(n)/n + \varepsilon$. Consequently, $\alpha \le \liminf_{n\to\infty} \frac{P(n)}{n}$ since $\varepsilon > 0$ was arbitrary, and so the required limit exists The proof of case (7) is similar.

Let a (resp., b) denote the limit formula in the definition of α_M (resp., β_M). So $\alpha_M = r + (p-r)a$ and $\beta_M = r + (p-r)b$. Clearly, $a \le \alpha \le b$. Suppose, for a contradiction, that $a < \alpha$. Let $0 < \varepsilon < (\alpha - a)/2$, and let the integers m_1, m_2 be large enough with $m_1 < m_2$ so that $\alpha \le \frac{P(m_2)}{m_2} \le \frac{P(m_1)}{m_1} \le \alpha + \varepsilon$ and $a - \varepsilon \le \frac{P(m_2)-P(m_1)}{m_2-m_1} \le a + \varepsilon$. Then

$$\alpha m_2 \le P(m_1) + P(m_2) - P(m_1) \le (\alpha+\varepsilon)m_1 + (m_2 - m_1)(a+\varepsilon).$$

It follows that $\alpha - a \leq \frac{m_1}{m_2 - m_1}\varepsilon$. Since m_2 can be chosen arbitrarily large compared to m_1, this leads to a contradiction. A similar argument also shows that $\alpha = b$. Thus $\alpha_M = r + (p-r)\alpha = \beta_M$. □

From Theorems 4.3 and 4.4 we obtain immediately

Corollary 4.5. *Let ℓ_M be a reflexive Orlicz sequence space, and suppose that its unit vector basis is upper or lower semi-homogeneous. Then the set of p's for which ℓ_p is isomorphic to a subspace of ℓ_M is a singleton.*

5. Further remarks

We end this note with some further observations and some natural problems suggested by this work.

Recall that two Banach spaces X, Y are said to have the same *linear dimension* if each of the spaces is isomorphic to a subspace of the other; then we write $\dim_l(X) = \dim_l(Y)$. It is well known that $C[0,1]$ and $C[0,1] \times \ell_1$ have equal linear dimension but they are not isomorphic [BM] (cf., [PW]). On the other hand, there are affirmative results in this direction for certain classes of Banach spaces. For instance, in nonseparable case a result due to Drewnowski [D] asserts that if X and Y possess uncountable symmetric bases then the condition $\dim_l(X) = \dim_l(Y)$ implies that the bases are equivalent. Here we indicate two classes of Banach spaces possessing the same property.

Theorem 5.1. *Let M_1, M_2 be two Orlicz functions satisfying the Δ_2-condition at zero, and assume that both the functions are simultaneously super- or sub-multiplicative. Then the following conditions are equivalent.*

(*i*) $\dim_l(\ell_{M_1}) = \dim_l(\ell_{M_2})$

(*ii*) *ℓ_{M_1} is isomorphic to ℓ_{M_2}.*

(*iii*) *The unit vector bases of ℓ_{M_1} and ℓ_{M_2} are equivalent.*

Proof. (*i*) *implies* (*ii*). Suppose that ℓ_{M_1} is isomorphic to a subspace of ℓ_{M_2} and vice versa. Then the unit vector basis (e_i) of ℓ_{M_1} is equivalent to a block basis of the unit vector basis (f_i) of ℓ_{M_2}, and vice versa (see [LT], p. 141). If the bases are upper or lower semi-homogeneous, then this means that (e_i) is dominated by (f_i), and (f_i) is dominated by (e_i). Thus they are equivalent.

The other implications are obvious. □

Remark 5.2. It is worth to note that a separable Orlicz space ℓ_M has upper (resp., lower) semi-homogeneous basis iff the basis dominates (resp., dominated by) the normalized blocks with constant coefficients. Equivalently, the unit vector basis of ℓ_M is upper semi-homogeneous iff M C-dominates every function in $E_{M,1}$ (see [LT], p. 140, for the definition of $E_{M,1}$). That is, there exists C such that

$$\frac{M(\lambda t)}{M(\lambda)} \leq CM(t) \text{ for all } 0 < \lambda, t < 1.$$

And it is lower semi-homogeneous iff M is C-dominated by every function in $E_{M,1}$.

Therefore, the following question arises naturally.

Problem 5.3. Let X be a Banach space with a symmetric basis (e_i). Suppose that (e_i) C-dominates (resp., C-dominated by) every normalized block sequence with constant coefficients. Is (e_i) upper (resp., lower) semi-homogeneous?

Remark 5.4. There are Banach spaces with a (even symmetric) semi-homogeneous basis which do not contain a copy of c_0 or ℓ_p. The unit vector bases of the Schlumprecht [Sch] and of Tzafriri [T] spaces are lower semi-homogeneous; see ([AS], remark before Proposition 2.5) and ([CS]; Proposition X.d.8, p. 113), respectively.

Remark 5.5. An Orlicz sequence space ℓ_M with a lower semi-homogeneous basis is either isomorphic to c_0 or ℓ_p or it contains uncountably many mutually non-equivalent symmetric basic sequences. This follows immediately from Theorem 3.3 in [S].

Remark 5.6. Recall that an Orlicz function M satisfying the Δ_2-condition at zero is called *minimal* (see [LT], p. 152) if $E_{N,1} = E_{M,1}$ for every $N \in E_{M,1}$. This simply means that if (e_i) is the unit vector basis of ℓ_M and (u_i) is a block basis with constant coefficients, then (u_i) has a further block basis (with constant coefficients with respect to (u_i)) which is equivalent to (e_i). We do not know if there exist a minimal Orlicz function M not equivalent to a power function so that ℓ_M admits a semi-homogeneous basis.

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