Banach Spaces and their Applications in Analysis, 77-82, de Gruyter 2007

ORLICZ SEQUENCE SPACES WITH DENUMERABLE SETS OF SYMMETRIC SEQUENCES

S. J. DILWORTH AND B. SARI

Abstract. We construct examples of Orlicz sequence spaces whose sets of symmetric sequences are, up to equivalence, countably infinite. This answers a question raised in [S].

1. Introduction

A question raised in [S] asks whether there exists an Orlicz sequence space such that the set of symmetric sequences in the space is, up to equivalence, precisely countably infinite. In this note we give an affirmative answer. In fact, for any countable ordinal γ we construct a reflexive Orlicz sequence space whose set of symmetric sequences with the domination order is, up to equivalence, order-isomorphic to the ordinal interval $[0, \gamma]$ in the reverse order. We also show that the converse holds when the set of symmetric sequences is countable and totally ordered in the domination order. In reflexive Orlicz sequence spaces the set of symmetric sequences coincides with the set of spreading models of the space, and the problem considered here is motivated by the study of the set of spreading models. We refer to [S] and [DOS] and the references therein for more details.

2. Preliminaries

An *Orlicz function* M is a real-valued continuous non-decreasing and convex function defined on $[0, 1]$ such that $M(0) = 0$ and $M(1) = 1$. For a given M, the *Orlicz sequence space* ℓ_M is the space of all sequences of scalars $x = (a_1, a_2, \dots)$ such that $\sum_{n=1}^{\infty} M(|a_n|/\rho) < \infty$ for some $\rho > 0$, equipped with the norm

$$\|x\| = \inf \{\rho > 0 : \sum_{n=1}^{\infty} M(|a_n|/\rho) \leq 1\}.$$

We will always assume that M satisfies the Δ_2-*condition* at zero (i.e., that there exists $C > 0$ such that $M(2t) \leq CM(t)$ for all $0 \leq t \leq 1/2$). Then the unit vectors form a normalized symmetric basis for ℓ_M. If N also satisfies the Δ_2-condition at zero then M *dominates* N, denoted by $N \leq M$, if there exists a constant $C > 0$ such that $N(t) \leq CM(t)$ for all $0 \leq t \leq 1$. For M and N satisfying the Δ_2-condition at zero $N \leq M$ if and only if the unit vector basis of ℓ_M dominates that of ℓ_N, that is, there exists a constant $C \geq 1$

such that for every finite sequence (a_i) of scalars,

$$\|\sum_i a_i e_i\|_{\ell_N} \leq C\|\sum_i a_i e_i\|_{\ell_M}.$$

We say that two such functions M and N are *equivalent* if $M \leq N$ and $N \leq M$. Thus M is equivalent to N if and only if the unit vector bases of ℓ_M and of ℓ_N are equivalent.

If M satisfies the Δ_2-condition at zero then an Orlicz sequence space ℓ_N is isomorphic to a subspace of ℓ_M if and only if N is equivalent to some function in $C_{M,1}$, where $C_{M,1}$ is the norm-closed convex hull in $C[0,1]$ of the set

$$E_{M,1} = \overline{\left\{\frac{M(\lambda t)}{M(\lambda)};\ 0 < \lambda < 1\right\}}. \tag{2.1}$$

See [LT, Lemma 4.a.6, remark (p. 141) and Theorem 4.a.8] for this result. Let (x_i) be a normalized symmetric (or even subsymmetric) sequence in ℓ_M. Then (x_i) is equivalent to the unit vector basis of ℓ_N, where $N \in C_{M,1}$ ([LT], Proposition 4.a.7 and Theorem 4.a.8). Thus there is a one-to-one correspondence between the set of symmetric sequences in ℓ_M and the Orlicz functions N in $C_{M,1}$. Moreover, these two sets are *order-isomorphic*. That is, if (x_i) and (y_i) are two normalized symmetric sequences in ℓ_M and N_1 and N_2 are the corresponding functions in $C_{M,1}$ respectively, then (y_i) dominates (x_i) if and only if $N_1 \leq N_2$.

We will use the following method of representing Orlicz functions by sequences of zeros and ones, introduced by Lindenstrauss and Tzafriri [LT, p. 161].

Fix $0 < \tau < 1$ and $1 < r < p < \infty$. For every sequence of zeros and ones, $\eta = (\eta(n))_{n=1}^{\infty}$ (i.e. $\eta(n) \in \{0,1\}$ for all n), let M_η be the piecewise linear function defined on $[0,1]$ satisfying $M_\eta(0) = 0$, $M_\eta(1) = 1$, and

$$M_\eta(\tau^k) = \tau^{rk+(p-r)\sum_{n=1}^{k}\eta(n)}, \qquad k = 1, 2, \ldots.$$

In our construction we will use the following particular class of such spaces as considered in [DOS, Lemma 2.3].

Lemma 2.1 (DOS)**.** *Let $1 < r < p < \infty$, and $0 < \tau < 1$. Let $(n_k) \subset \{1, 2, 2^2, \ldots\}$ with $n_1 = 1$ and put*

$$\rho(i) = \begin{cases} 0 & \text{if } i = n_k \\ 1 & \text{otherwise.} \end{cases}$$

Let $M := M_\rho$ be the corresponding Orlicz function. Then ℓ_M is reflexive and every symmetric sequence in ℓ_M is equivalent to either the unit vector basis of ℓ_M or to the unit vector basis of ℓ_p.

3. The Construction

Theorem 3.1. *For every countable ordinal $\gamma \geq 0$ there exists a reflexive Orlicz sequence space ℓ_M such that the set of normalized symmetric sequences in ℓ_M is order-isomorphic to $[0, \gamma]$ with the reverse order.*

Proof. The first part of the construction bears similarities to the one in [DOS, Theorem 3.1]. For every $\alpha < \gamma$ we construct a sequence ρ_α of the form given in Lemma 2.1 so that the collection of corresponding Orlicz functions $(M_{\rho_\alpha})_{\alpha<\gamma}$ (for the same τ, r and p) is order-isomorphic to $[0, \gamma] \setminus \{\gamma\}$ with the reverse order. Then we patch the ρ_α sequences together into one sequence ϕ in such a way that $C_{M_\phi,1}$ of the corresponding Orlicz function M_ϕ (again with the same τ, r and p) will consist, up to equivalence, only of the collection $\{M_\phi\} \cup \{M_{\rho_\alpha}\}_{\alpha<\gamma} \cup \{t^p\}$. Thus ℓ_{M_ϕ} will be the desired Orlicz sequence space.

The ρ_α sequences will be constructed simultaneously on a finite interval at a time using a variation on what is called the *ε-domination procedure* in [DOS], which we recall now for completeness.

Let (ρ_j) be a collection of sequences to be constructed, and let $A \subset \mathbb{N}$ and $\varepsilon > 0$. Put $\rho_j(1) = 0$ for all j, and suppose that all the sequences are already defined on $[1, N]$.

The (ε, A)-domination procedure extends the definition of the ρ_j's to an initial segment $[1, N_1]$ for some $N_1 > N$.

Suppose that both A and $\mathbb{N} \setminus A$ are non-empty. Choose a sufficiently large (just how large is specified below) integer $m > N$. For all $k \in \mathbb{N} \setminus A$ place 0's on the coordinates from $[N+1, m] \cap \{2^n : n = 0, 1, 2, \ldots\}$ of the ρ_k's (while the rest of the coordinates of the interval are filled with 1's), and for all $j \in A$ place 1's on *all* the coordinates from $[N + 1, m]$ of the ρ_j's , where m is chosen so that

$$\sum_{i=1}^{m} \rho_j(i) - \sum_{i=1}^{m} \rho_k(i)$$

is sufficiently large to ensure that

$$\frac{M_{\rho_j}(\tau^m)}{M_{\rho_k}(\tau^m)} < \varepsilon, \text{ for all } j \in A,\ k \in \mathbb{N} \setminus A.$$

We now pass to the main construction. Let γ be a countable ordinal. For simplicity we shall assume that γ is a limit ordinal (the argument in the successor ordinal case is similar). Let $L = [0, \gamma] \setminus \{\gamma\}$ with the reverse order. Note that every subset of L has a *maximum* element in this order. Let $(\alpha_j)_{j=1}^{\infty}$ be an enumeration of L.

For every $j \in \mathbb{N}$ and every $\varepsilon_k = 2^{-k}$, $k = 1, 2, \ldots,$ we carry out an (ε_k, A_j)-domination procedure for $A_j = \{i \in \mathbb{N} : \alpha_i < \alpha_j \text{ in } L \text{ order}\}$ as described above. Note that A_j and $\mathbb{N} \setminus A_j$ are both nonempty. We shall call α_j the *dominating node* and we shall say that α_i

is ε-*dominated* by α_j if $i \in A_j$. Since there are countably many choices we can enumerate some order in which to carry out all (ε_k, A_j)-dominations.

At the end we obtain sequences ρ_j for each $\alpha_j \in L$ with the following easily checked properties:

(i) Each ρ_j begins with 0;

(ii) The zeros occur only on coordinates (2^k);

(iii) $\alpha_i \geq \alpha_j$ (in the order of L) if and only if $\rho_i(n) \leq \rho_j(n)$ for all n (we say that ρ_j dominates ρ_i *pointwise*). Thus in this case $M_{\rho_j} \leq M_{\rho_i}$;

(iv) L is order-isomorphic to the collection (M_{ρ_j}).

Now let ϕ be the sequence obtained by writing out the initial segments of the ρ_j's with extra 1's so that each initial segment of length n is followed by n 1's before the next initial segment starts:

$$\phi = (\rho[1], 1, \rho_1[2], 1, 1, \rho_2[2], 1, 1, \rho_1[3], 1, 1, 1, \rho_2[3], 1, 1, 1, \rho_3[3], 1, 1, 1, \rho_1[4], \ldots),$$

where $\rho[n]$ denotes the initial segment of ρ of size n.

We will show that $M := M_\phi$ is the desired Orlicz function. Recall that $N \in E_{M_\rho,1}$ if and only if $N = N_\psi$ where ψ is a pointwise limit of $(T^m\phi)_{m\in\mathbb{N}}$ [LT, p. 161]. From the form of ϕ it is clear that the M_{ρ_j} functions and t^p, which is represented by the sequence $(1, 1, 1, \ldots)$, belong to $E_{M,1}$, and, moreover, that M_ϕ strictly dominates each M_{ρ_j}.

We show in the next lemma that, together with M_ϕ, these are the only elements of $E_{M,1}$ up to equivalence.

For $n \geq 0$ by $T^n\rho$ denote the sequence obtained by shifting ρ to the left n coordinates, and for $n < 0$ by $T^n\rho$ denote the sequence ρ with n 1's put at the beginning.

One easily verifies from the definition of $\rho_\alpha := \rho_j$ that $M_{\rho_\alpha}(t) \geq M_{T^m(\rho_\alpha)}(t)$ for all $\alpha = \alpha_j \in L$, $m \in \mathbb{Z}$, and $t \in [0, 1]$. This fact will be used in the proof of our main result.

Lemma 3.2. *The pointwise limits of* $(T^m\phi)_{m\in\mathbb{N}}$ *are of the form*

(a) all sequences $T^m\rho_j$ *for* $\alpha_j \in L$ *and* $m \in \mathbb{Z}$*, or*

(b) all sequences with at most one term equal to zero and all other terms equal to one.

Proof of Lemma 3.2. Let ψ be such a limit. If ψ is not of type (b) then there is a fixed $m_0 \in \mathbb{Z}$ so that ψ is a pointwise limit of a sequence of sequences of the form $(T^{m_0}\rho_{j_k})_{k\geq1}$. For each $i \in \mathbb{N}$, let $\alpha_{j(i)}$ be the greatest element in L such that $T^{m_0}\rho_{j(i)}$ agrees with ψ on the first i coordinates. (Every subset of L has a maximum, so $\alpha_{j(i)}$ exists.) Clearly, $\alpha_{j(1)} \geq \alpha_{j(2)} \geq \alpha_{j(3)} \geq \ldots$. Let $\alpha = \lim_i \alpha_{j(i)}$. The limit exists in $L \cup \{\gamma\}$ since $L \cup \{\gamma\}$ is an ordinal in the reverse order.

If $\alpha > \gamma$, then we claim that $\psi = T^{m_0}\rho_\alpha$. Indeed, clearly $T^{m_0}\rho_\alpha \geq \psi$ pointwise since $\rho_\alpha \geq \rho_{j(i)}$ pointwise for all i (by construction). But consider a subinterval of $\mathbb{N}$ on which α

is ε-dominated by some β with $\beta > \alpha$. Since $\alpha_{j(i)} \searrow \alpha$ we have $\beta > \alpha_{j(i)}$ for all sufficiently large i. Hence $\alpha_{j(i)}$ is also ε-dominated by β for all sufficiently large i. This proves that $\psi \geq T^{m_0}\rho_\alpha$ pointwise.

If $\alpha = \gamma$ then a similar argument shows that ψ agrees with an ε-dominated sequence on every subinterval of $\mathbb{N}$ on which an ε-domination takes place, and hence ψ is of type (b). □

It remains to show that every function in $C_{M,1}$ is equivalent to M_ϕ or to one of the M_{ρ_j}'s or to t^p.

Let $M_\sigma \in C_{M,1}$. Then by Lemma 3.2 M_σ is equivalent either to M_ϕ or to a uniform limit of a sequence (M_{σ_n}) of finite convex combinations of functions in $E_{M,1}$ of the form

$$M_{\sigma_n} = \sum_{m\in\mathbb{Z},\alpha\geq\gamma} \lambda^n_{m,\alpha} M_{T^m\rho_\alpha},$$

where the positive 'mass' coefficients $\lambda^n_{m,\alpha}$ sum to 1 for each n. Here $\rho_\gamma := (1,1,1,\ldots)$ (so M_{ρ_γ} is equivalent to t^p) represents the minimum symmetric sequence in the domination order.

Define the following function ξ on L:

$$\xi(\alpha) = \lim_{m\to\infty} \limsup_{n\to\infty} \sum_{\beta\geq\alpha,|j|\leq m} \lambda^n_{j,\beta}.$$

We now consider two cases.

In the first case, suppose that ξ is not identically zero. Then there exists a greatest element α_0 with $\xi(\alpha_0) > 0$ (since every subset of L has a maximum element). This implies that there exists $m_0 \in \mathbb{N}$ with

$$\limsup_{n\to\infty} \sum_{\beta\geq\alpha_0,|j|\leq m_0} \lambda^n_{j,\beta} := \delta > 0.$$

We claim that in this case M_σ is equivalent to $M_{\rho_{\alpha_0}}$. Clearly, M_σ dominates $M_{\rho_{\alpha_0}}$ since $\delta > 0$ and $M_{\rho_\beta} \geq M_{\rho_\alpha}$ whenever $\beta \geq \alpha$. Consider a fixed initial segment of $\mathbb{N}$ on which α_0 is ε-dominated at least once. Then the collection of dominating nodes which ε-dominate α_0 on this initial segment is finite and non-empty. So there is a least (in the ordering of L) dominating node $\beta_0 > \alpha_0$ from this collection.

Thus $\xi(\beta_0) = 0$ by the definition of α_0. This means that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{\beta\geq\beta_0,|j|\leq m} \lambda^n_{j,\beta} = 0.$$

Note that if $|m| > C$, where C depends only on the fixed initial segment of $\mathbb{N}$, then for all $\alpha \in L$, $T^m\rho_\alpha$ will contain at most one zero in that initial segment. But the sequences $T^m\rho_\alpha$ for $\alpha \geq \beta_0$ and $m \leq C$ which occur in the definition of M_{σ_n} are assigned a 'total mass' which tends to zero as $n \to \infty$, and hence their contribution vanishes in the limit. On the other hand, if $\alpha < \beta_0$, then on this initial segment α is ε-dominated whenever α_0

is ε-dominated, and hence ρ_{α_0} is dominated pointwise on this initial segment by ρ_α. Recall that $M_\alpha(t) \geq M_{T^m(\alpha)}(t)$ for all $\alpha \in L$, $m \in \mathbb{Z}$, and $t \in [0,1]$. Taking the limit as $n \to \infty$ it follows that $M_\sigma \leq M_{\rho_{\alpha_0}}$.

In the second case, suppose that $\xi(\alpha) = 0$ for all α. This easily implies that M_σ is equivalent to the minimum element $M_{\rho(\gamma)}$ (which is equivalent to t^p).

Thus, we have proved that $\{\ell_{M_\phi}\} \cup \{\ell_{M_{\rho_j}}\}_{j=1}^\infty \cup \{\ell_p\}$ is the set of symmetric sequences in ℓ_{M_ϕ} up to equivalence. In the domination order, ℓ_{M_ϕ} is the maximum element and ℓ_p is the minimum element, and hence the set is order-isomorphic to the ordinal sum $1 + \gamma^+$ with the reverse order, where γ^+ denotes the successor ordinal to γ, which in turn is isomorphic to the ordinal interval $[0, \gamma]$ with the reverse order. □

Corollary 3.3. *Suppose that L is a countable totally ordered set. Then there exists a reflexive Orlicz space ℓ_M such that the collection of symmetric sequences in ℓ_M with the domination order is order-isomorphic to L if and only if L is order-isomorphic to an ordinal interval $[0, \gamma]$ in the reverse order for some countable ordinal γ.*

Proof. Sufficiency follows from our main result. Conversely, if L is the set of symmetric sequences up to equivalence in ℓ_M, where ℓ_M is reflexive, then L coincides with the set of spreading models of ℓ_M up to equivalence [S], and if L is countable and totally ordered, then L is well-ordered in the reverse order [DOS, Theorem 3.7], and moreover L has a minimum element [S]. Hence L is order-isomorphic to $[0, \gamma]$ in the reverse order for some countable ordinal γ. □

References

[DOS] S. J. Dilworth, E. Odell and B. Sarı, *Lattice structures and spreading models*, Israel J. Math, to appear.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces vol. I: Sequence spaces*, Springer-Verlag, New York, 1977.

[S] B. Sarı, *On the structure of the set of symmetric sequences in Orlicz sequence spaces*, Canadian Math. Bull., to appear.

Department of Mathematics, University of South Carolina, Columbia, SC 29208-0001
E-mail address: dilworth@math.sc.edu

Department of Mathematics, University of North Texas, Denton, TX 76203-1430
E-mail address: bunyamin@unt.edu