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LATTICE STRUCTURES AND SPREADING MODELS

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ABSTRACT

We consider problems concerning the partial order structure of the set of spreading models of Banach spaces. We construct examples of spaces showing that the possible structure of these sets include certain classes of finite semi-lattices and countable lattices and all finite lattices.

Contents

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0. Introduction

The spreading models of a Banach space X usually have a simpler and better structure, both individually and collectively, than the class of subspaces of X. Sometimes knowledge of the spreading models can be used to deduce subspace knowledge about X itself (e.g., [AOST, OS1]) but the relationship is still not completely understood. Spaces with no "nice" subspaces can have very nice spreading models (e.g., [AD]).

In this paper we further explore the relationship between a space and its spreading models. In particular, we study the possible partial order structures of the spreading models of X generated by normalized weakly null sequences. In §1 we recall what is known and unknown and present some new structural observations along with the relevant background. In §2 and §3 we construct spaces X with certain prescribed spreading model structures. In §2 we construct for each $n \in \mathbb{N}$, a space X_n with $(SP_w(X_n), \leq)$ order isomorphic to $(\mathcal{P}(n) \setminus \{\emptyset\}, \subseteq)$ where $\mathcal{P}(n)$ is the power set of $\{1, \ldots, n\}$. In §3 we show that if L is a countable lattice with a minimum element not containing an infinite strictly increasing sequence, then there exists a reflexive space X_L with $(SP_w(X_L), \leq)$ order-isomorphic to L. The construction uses some beautiful classical work of Lindenstrauss and Tzafriri [LT] on Orlicz sequence spaces.

1. Background, questions and observations

We use standard Banach space notation and terminology as in [LT].

Let X be a separable infinite-dimensional Banach space. A normalized basic sequence $(x_i) \subseteq X$ generates a **spreading model** $(\tilde{x}_i)$ if for some $\varepsilon_n \downarrow 0$ for all $n \in \mathbb{N}$ and $(a_i)_1^n \subseteq [-1, 1]$,

$$(1+\varepsilon_n)^{-1}\Big\|\sum_{i=1}^n a_i\tilde{x}_i\Big\| \leq \Big\|\sum_{i=1}^n a_i x_{k_i}\Big\| \leq (1+\varepsilon_n)\Big\|\sum_{i=1}^n a_i\tilde{x}_i\Big\|$$

for all $n \leq k_1 < \cdots < k_n$. $(\tilde{x}_i)$ is a basic sequence which is 1-spreading and suppression-1 unconditional if (x_i) is weakly null. Every normalized basic sequence has a subsequence which generates a spreading model (see [BL] for these and more elementary facts about spreading models).

We let $[(\tilde{x}_i)]$ denote the equivalence class of all spreading models of X which are equivalent (see below) to $(\tilde{x}_i)$. $SP_w(X)$ denotes the set of all such $[(\tilde{x}_i)]$ where we restrict ourselves only to spreading models generated by weakly null sequences. If $SP_w(X) = \emptyset$ then X is a Schur space, so every normalized spreading model of X is equivalent to the unit vector basis of ℓ_1 by Rosenthal's ℓ_1

theorem [R].

If $[(\tilde{x}_i)], [(\tilde{y}_i)] \in SP_w(X)$ we write $[(\tilde{x}_i)] \le [(\tilde{y}_i)]$ if for some $C < \infty$, $(\tilde{y}_i)$ **C-dominates** $(\tilde{x}_i)$, i.e., for all $(a_i) \subseteq \mathbb{R}$

$$\Big\| \sum a_i \tilde{x}_i \Big\| \le C \Big\| \sum a_i \tilde{y}_i \Big\|.$$

$(\tilde{x}_i)$ and $(\tilde{y}_i)$ are **equivalent** if each dominates the other. $(SP_w(X), \le)$ is a partially ordered set.

Occasionally we will consider a specific $(\tilde{x}_i)$ and shall abuse notation by writing "let $(\tilde{x}_i) \in SP_w(X)$." c_{00} denotes the linear space of finitely supported real sequences.

Fact 1.1 ([AOST]): $(SP_w(X), \le)$ is a **semi-lattice**, i.e., each pair of elements of $SP_w(X)$ admit a least upper bound. Moreover, if $(\tilde{x}_i), (\tilde{y}_i) \in SP_w(X)$ there exists $(\tilde{z}_i) \in SP_w(X)$ which is 2-equivalent to the subsymmetric norm on c_{00} given by

$$\|(a_i)\| = \Big\| \sum a_i \tilde{x}_i \Big\| \vee \Big\| \sum a_i \tilde{y}_i \Big\|.$$

Fact 1.2 ([AOST]): Every countable subset of $(SP_w(X), \le)$ admits an upper bound. Moreover, if $(\tilde{x}_i^n)_{i=1}^\infty \in SP_w(X)$ for $n \in \mathbb{N}$ and $(C_n)_{n=1}^\infty \subseteq (1, \infty)$ with $\sum_{n=1}^\infty C_n^{-1} \le 1$, then there exists $(\tilde{z}_i) \in SP_w(X)$ which C_n-dominates $(\tilde{x}_i^n)$ for each $n \in \mathbb{N}$. In addition for $(a_i) \in c_{00}$

$$\Big\| \sum a_i \tilde{z}_i \Big\| \le C_1 \bigg(\sum_{n=1}^\infty C_n^{-1} \Big\| \sum a_i \tilde{x}_i^n \Big\| \bigg).$$

We shall designate this $(\tilde{z}_i)$ by the notation $(\tilde{z}_i) = (\sum C_n^{-1} \tilde{x}_i^n)$, which in fact is motivated by the proof in [AOST] (the precise quantification as given above is noted in [S2]).

Fact 1.3 ([S2]): If $SP_w(X)$ admits an infinite strictly increasing sequence, then $SP_w(X)$ is uncountable. In fact, there exist $[(\tilde{y}_i^\alpha)] \in SP_w(X)$ for $\alpha < \omega_1$ so that $[(\tilde{y}_i^\alpha)] < [(\tilde{y}_i^\beta)]$ if $\alpha < \beta < \omega_1$.

Our next result is motivated by the proof of Fact 1.3.

Theorem 1.4: *Let I be an infinite set and let $(\tilde{x}_i^\alpha)_{i=1}^\infty \in SP_w(X)$ for $\alpha \in I$. For $A \subseteq I$ define a subsymmetric norm on c_{00} by $R_A(a_i) = \sup_{\alpha \in A} \| \sum a_i \tilde{x}_i^\alpha \|$. If for every non-empty finite $F \subseteq I$, R_I is not equivalent to R_F, then $SP_w(X)$ admits an infinite strictly increasing sequence.*

Proof: We may assume $I=\mathbb{N}$. We shall construct a strictly increasing sequence $(\tilde{y}_i^n)_{i=1}^\infty$ for $n\in\mathbb{N}$. We let

$$(\tilde{y}_i^1)=\left(\sum_{n=1}^\infty 2^{-n}\tilde{z}_i^n\right)\quad\text{(see Fact 1.2)}$$

where $(\tilde{z}^n)_{n=1}^\infty$ is a reordering of $(\tilde{x}^n)_{n=1}^\infty$ selected as follows. Let $\varepsilon_n\downarrow 0$ and for each $n\in\mathbb{N}$, $\tilde{z}^{2n}$ is chosen so that for some $(a_\ell^n)_{\ell=1}^\infty\in c_{00}$,

$$R_I(a_\ell^n)_{\ell=1}^\infty=1,\qquad \Big\|\sum_\ell a_\ell^n\tilde{z}_\ell^{2n}\Big\|>\frac{1}{2}$$

and $R_{I_n}(a_\ell^n)_{\ell=1}^\infty<\varepsilon_n 2^{-2n}$ where

$$I_n=\{m\in\mathbb{N}:\tilde{x}^m=\tilde{z}^j\text{ for some } j\le 2n-1\}.$$

$\tilde{z}^n$ for n odd is selected arbitrarily so as to exhaust the collection $(\tilde{x}^s)_{s\in\mathbb{N}}$.

For $n\in\mathbb{N}$ we have (see Fact 1.2)

$$\begin{aligned}\Big\|\sum_\ell a_\ell^n\tilde{y}_\ell^1\Big\| &\le 2R_{I_n}(a_\ell^n)_{\ell=1}^\infty+2\cdot 2^{-2n}\Big\|\sum_\ell a_\ell^n\tilde{z}_\ell^{2n}\Big\|+2\sum_{m>2n}2^{-m}\Big\|\sum_\ell a_\ell^n\tilde{z}_\ell^m\Big\|\\ &<2\varepsilon_n 2^{-2n}+2\cdot 2^{-2n}+2\sum_{m>2n}2^{-m}\\ &=(2\varepsilon_n+4)2^{-2n}.\end{aligned}$$

Furthermore,

$$\Big\|\sum a_\ell^n\tilde{y}_\ell^1\Big\|\ge 2^{-2n}\Big\|\sum_\ell a_\ell^n\tilde{z}_\ell^{2n}\Big\|>\frac{1}{2}2^{-2n}.$$

Thus we obtain that $(\tilde{x}_i^n)<(\tilde{y}_i^1)$ for all $n\in\mathbb{N}$ and $(\tilde{y}_i^1)<R_I$. Moreover, we can iterate the argument beginning anew with the collection $\{(\tilde{y}_i^1)\}\cup\{(\tilde{x}_i^n)\}_{n=1}^\infty$, which satisfies the same hypothesis as $\{(\tilde{x}_i^n)\}_{n=1}^\infty$, to obtain $\tilde{y}^2$, and so on. ∎

Fact 1.5: $SP_w(X)$ can be hereditarily uncountable [AOST], i.e., $SP_w(Y)$ is uncountable for all infinite-dimensional subspaces Y of X. If $SP_w(X)$ is countable, then by a diagonal argument one can find $X_0\subseteq X$ with $SP_w(X_0)=SP_w(Y)$ for all $Y\subseteq X_0$. It may be that then $|SP_w(X_0)|=1$ but this remains open.

We also have the

PROBLEM 1.6: *If X is reflexive and $SP_w(X)$ is countable; must some $(\tilde{x}_i) \in SP_w(X)$ be equivalent to the unit vector basis of c_0 or ℓ_p for some $1 \le p < \infty$?*

If so, this would be a case where one would have a stronger theorem than Krivine's [K]. Not every reflexive space has a spreading model isomorphic to c_0 or some ℓ_p ([OS2], [AOST]). In the non-reflexive case it is possible to have $|SP_w(X)| = 1$ yet the unique spreading model is not c_0 or any ℓ_p. This is the case for certain Lorentz sequence spaces $d_{w,1}$ (see §2).

Problem 1.6 was raised and partially solved in the case $|SP_w(X)| = 1$ in [AOST]. We give some further partial results below.

Remark 1.7: Assume that $SP_w(X)$ is countable or, more generally, does not admit an infinite strictly increasing sequence. For $[(\tilde{x}_i)] \in SP_w(X)$ and $(a_i) \in c_{00}$, define

$$R(a_i) \equiv R_{[(\tilde{x}_i)]}(a_i) = \sup \left\{ \left\| \sum_1^n a_i \tilde{y}_i \right\| : (\tilde{y}_i) \in [(\tilde{x}_i)] \right\}.$$

By Theorem 1.4 R is equivalent to $(\tilde{x}_i)$. Thus for each $(\tilde{y}_i) \in [(\tilde{x}_i)]$ there exists $C < \infty$ so that $(\tilde{y}_i)$ C-dominates every $(\tilde{z}_i) \in [(\tilde{x}_i)]$. Also, by [S2], there exists $p = p(\tilde{x}_i) \in [1, \infty]$, so that for every $1 \le q < p$ there exists $C_q < \infty$, so that for all $(a_i) \subseteq \mathbb{R}$,

$$\left(\sum |a_i|^p \right)^{1/p} \le R(a_i) \le C_q \left(\sum |a_i|^q \right)^{1/q}.$$

$p(\tilde{x}_i)$ is the infimum of the "Krivine p's" for $(\tilde{x}_i)$ (see [S2]). It is mistakenly stated in [S2] that in this case, $p(\tilde{x}_i)$ is the only Krivine p. However, this is not yet clear.

Remark 1.8: Let $SP_w(X)$ be stabilized hereditarily for X. Then for all $(\tilde{x}_i) \in SP_w(X)$ there exist $X_0 \subseteq X$ and $C < \infty$ such that: for all $Y \subseteq X_0$ there exists $(\tilde{y}_i) \in SP_w(Y)$ which is C-equivalent to $(\tilde{x}_i)$.

The proof is elementary. Assume it is not true, and use a diagonal argument to get a contradiction.

THEOREM 1.9: *Suppose that $SP_w(X)$ is countable and that $SP_w(Y^*)$ is countable for all infinite-dimensional subspaces Y of X. Then every $(\tilde{e}_i) \in SP_w(X)$ is equivalent to the unit vector basis of c_0 or ℓ_p for some $1 \le p < \infty$.*

Proof: Let (e_i) be a normalized weakly null sequence in X generating the spreading model $(\tilde{e}_i)$. By passing to a subsequence and renorming we may assume that (e_i) is bimonotone basic and Schreier-unconditional, i.e., for some $\varepsilon_n \downarrow 0$ and all $F \in S_1$ and $(a_i) \in c_{00}$,

$$\Big\|\sum_{i\in F} a_i e_i\Big\| \le (1+\varepsilon_{\min F})\Big\|\sum a_i e_i\Big\|, \tag{1.1}$$

(see [O1]). Here $F \in S_1$ (first Schreier class) if $|F| \le \min F$.

We may assume that no subsequence of (e_i) is equivalent to the unit vector basis of c_0. Thus by passing to a further subsequence we may assume that (f_i), the sequence of biorthogonal functions to (e_i), is weakly null in $[(e_i)]^*$ [O2, Cor. 4.4]. From (1.1) it is easy see that (f_i) is normalized and has spreading model $(\tilde{f}_i)$ which is 1-equivalent to $(\tilde{e}_i^*)$, the biorthogonal functionals to $(\tilde{e}_i)$ in $[(\tilde{e}_i)]^*$. By Krivine's theorem [K] it suffices to prove that, for some $D < \infty$, every spreading model $(\tilde{x}_i)$ of an identically distributed block basis (x_i) of (e_i) with $\|x_i\| \to 1$, is D-equivalent to $(\tilde{e}_i)$. Note that (x_i) is weakly null and $(\tilde{x}_i)$ is equivalent to an identically distributed normalized block basis of $(\tilde{e}_i)$ and hence to $(\tilde{e}_i)$. Since $SP_w(X)$ is countable, by Theorem 1.4 there exists $C_1 < \infty$ (which depends only on $(\tilde{e}_i)$) such that

$$\Big\|\sum a_i\tilde{x}_i\Big\| \le C_1\Big\|\sum a_i\tilde{e}_i\Big\| \quad ((a_i)\in c_{00}). \tag{1.2}$$

We may choose an identically distributed block basis (g_i) of (f_i) with $\mathrm{supp}(g_i) \subseteq \mathrm{supp}(x_i)$, $\|g_i\| \to 1$, and $g_i(x_i) \to 1$. Note that (g_i) has spreading model $(\tilde{g}_i)$ which is 1-equivalent to an identically distributed block basis of $(\tilde{f}_i)$. Also (g_i) is weakly null and since $SP_w(Y^*)$, where $Y = [(e_i)]$, is countable we have, again by Theorem 1.4, that there exists $C_2 < \infty$ (which depends only on $(\tilde{f}_i)$) such that

$$\Big\|\sum a_i\tilde{g}_i\Big\| \le C_2\Big\|\sum a_i\tilde{f}_i\Big\| \quad ((a_i)\in c_{00}).$$

Let h_i be the restriction of g_i to $[(x_i)]$. Since (x_i) is bimonotone and Schreier unconditional (by passing to a subsequence if necessary), we have, as above, that (h_i) has a spreading model $(\tilde{h}_i)$ in $[(x_i)]^*$ which is 1-equivalent to $(\tilde{x}_i^*)$, the biorthogonal functionals to $(\tilde{x}_i)$. Thus for $(a_i) \in c_{00}$,

$$\Big\|\sum a_i\tilde{x}_i^*\Big\| = \Big\|\sum a_i\tilde{h}_i\Big\| \le \Big\|\sum a_i\tilde{g}_i\Big\| \le C_2\Big\|\sum a_i\tilde{f}_i\Big\|.$$

By duality,

$$\Big\|\sum a_i\tilde{x}_i\Big\| \ge \frac{1}{C_2}\Big\|\sum a_i\tilde{e}_i\Big\| \quad ((a_i)\in c_{00}). \tag{1.3}$$

Thus by (1.2) and (1.3), $(\tilde{x}_i)$ is $D \equiv C_1C_2$-equivalent to $(\tilde{e}_i)$. ∎

THEOREM 1.10: *Let X be reflexive with $|SP_w(X)| = |SP_w(X^*)| = 1$. Assume also that the element of $SP_w(X^*)$ is equivalent to the biorthogonal functionals of the element $(\tilde{x}_i)$ in $SP_w(X)$. Then $(\tilde{x}_i)$ is equivalent to the unit vector basis of c_0 or ℓ_p for some $1 \le p < \infty$.*

Proof: First we note that if X_0 is any infinite-dimensional subspace of X, then X_0 satisfies the same hypothesis as X. Indeed, the only question here is the uniqueness of the spreading models in X_0^*. Let $(\tilde{f}_i)$ be a normalized spreading model for X_0^* generated by (f_i). Then (f_i) is the image under the quotient map of a seminormalized weakly null sequence in X^* and this yields that $(\tilde{x}_i^*)$ dominates $(\tilde{f}_i)$. A similar argument applied to the sequence biorthogonal to (f_i) shows that $(\tilde{f}_i)$ dominates $(\tilde{x}_i^*)$. The result now follows from Theorem 1.9. ∎

The proof of Theorem 1.9 contains the following result.

THEOREM 1.11: *Let (e_i) be a normalized basis for a reflexive space X which is C-Schreier unconditional for some $C < \infty$, i.e.,*

$$\Big\|\sum_F a_i e_i\Big\| \le C \Big\|\sum a_i e_i\Big\| \quad \textit{for all } F \in S_1$$

and $(a_i) \subseteq \mathbb{R}$. If $|SP_w(X)| = |SP_w(X^*)| = 1$, then the unique spreading model of X is equivalent to the unit vector basis of c_0 or ℓ_p for some $1 \le p < \infty$.

Remark 1.12: If $SP_w(X)$ is countably infinite then $SP_w(X)$ contains

$$\{(\tilde{x}_i^n)_{i=1}^\infty : n \in \mathbb{N}\}$$

with either $(\tilde{x}_i^n) > (\tilde{x}_i^m)$ for all $n < m$ or $(\tilde{x}_i^n)$ and $(\tilde{x}_i^m)$ mutually incomparable for all $n \ne m$. Indeed Ramsey's theorem yields a subsequence of any sequence of spreading models satisfying either one of the two possibilities above or a sequence that is strictly increasing. The latter is ruled out by Fact 1.3. Both possibilities can occur for reflexive spaces. As noted elsewhere [AOST] (see also Theorem 3.7 below), it is easy to check that every spreading model of $(\sum \oplus \ell_{p_n})_{p_1}$ is equivalent to some ℓ_{p_n} if $p_1 < p_2 < \cdots$. In §3 we shall show the second (mutually incomparable) possibility.

The uncountable case is less clear.

Problem 1.13: *If $SP_w(X)$ is uncountable must there exist $\{(\tilde{x}_i^\alpha)_{i=1}^\infty : \alpha < \omega_1\} \subseteq SP_w(X)$ which is either strictly increasing w.r.t. α, strictly decreasing or consists of mutually incomparable elements?*

If there is a counterexample, say, X to this question, then by Fact 1.3, $SP_w(X)$ cannot contain an infinite increasing sequence. However, we do not know the answer to this generalized version of Problem 1.13.

Problem 1.14: *Let L be an uncountable semi-lattice which admits no infinite strictly increasing sequence. Must L admit a family $(x_\alpha)_{\alpha<\omega_1}$ with either*
(i) *$\forall \alpha \neq \beta$, x_α and x_β are incomparable or*
(ii) *$\forall \alpha < \beta < \omega_1$, $x_\alpha > x_\beta$?*

We are indebted to S. Todorcevic for the following remarks concerning problems 1.13 and 1.14. First, the lattice version of Problem 1.13 has a negative answer. Indeed, one may take C to be an uncountable linear ordering whose cartesian square C^2 (ordered by $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$) can be covered by countably many chains [T1]. Then C^2 is an uncountable lattice with no uncountable chains nor with increasing or decreasing copies of ω_1.

Second, let P be the set of all non-decreasing maps on $\mathbb{N}$. We let $f \leq g$ if $f(n) \leq g(n)$ for all n and $f \leq^* g$ if for some m, $f(n) \leq g(n)$ for all $n \geq m$. $(P, \leq)$ is a lattice. Let A be any subset of P that forms a well-ordered and unbounded chain of some regular order type θ ($\theta = \omega_1$ if (CH) holds) relative to $\leq^*$. Then it follows from Theorem 1.1 of [T2] that A has no uncountable antichains of cardinality θ under $\leq$. Thus, assuming (CH), if we let $[A]$ be the lattice generated by A (under $\leq$), then $[A]$ is well-founded, it has no uncountable antichains, and obviously it contains no copies of ω_1.

Our work in the next two sections suggests the following.

Problem 1.15: *Let L be a countable semi-lattice not admitting an infinite strictly increasing sequence. Does there exist an X (possibly even reflexive) with $(SP_w(X), \leq)$ order-isomorphic to L?*

2. Spreading model sets without a minimum element

In this section we shall construct some families of Banach spaces whose spreading model sets do not have a minimum element in the domination ordering. The Banach spaces in question are finite direct sums of certain Lorentz sequence spaces $d(w, p)$.

The construction depends on the existence of an arbitrary number of incomparable submultiplicative functions. We begin with a technical definition to facilitate the discussion.

Definition 2.1: Let $2 \leq n_0 \leq \infty$ and let S be a real-valued function defined on $[1, n_0]$. We shall say that S is **submultiplicative on** $[1, n_0]$ (or on $[1, \infty)$ if $n_0 = \infty$) if S satisfies the following conditions:

(a) S is piecewise-linear, continuous, strictly increasing and concave.

(b) $S(x) = x$ for $1 \leq x \leq 2$.

(c) $S(xy) \leq S(x)S(y)$ for all x, y such that $1 \leq x, y, xy \leq n_0$.

LEMMA 2.2: *Suppose that $2 \leq n_0 < \infty$ and that S is submultiplicative on $[1, n_0]$. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the extension S_ε of S to the interval $[1, n_0^2]$ defined by*

$$S_\varepsilon(x) = \begin{cases} S(x) & \text{for } 1 \leq x \leq n_0 \\ S(n_0) + \varepsilon(x - n_0) & \text{for } n_0 < x \leq n_0^2 \end{cases}$$

is submultiplicative on $[1, n_0^2]$.

Proof: Since S is continuous, piecewise-linear, and strictly increasing on $[1, n_0]$, there exists $c > 0$ such that

$$S(x) \geq S(x-h) + ch \quad (1 \leq x - h \leq x \leq n_0). \tag{2.1}$$

Define $\tilde{S}$ on $[1, n_0^2]$ as follows:

$$\tilde{S}(x) := \inf\{S(a)S(b) \colon x = ab, 1 \leq a, b \leq n_0\}.$$

Since S is continuous and strictly increasing on $[1, n_0]$ it follows that $\tilde{S}$ is continuous and strictly increasing on $[1, n_0^2]$. Moreover, conditions (b) and (c) of Definition 2.1 imply that $\tilde{S}(x) = S(x)$ for all $x \in [1, n_0]$. Suppose that $x := n_0 + h$ satisfies $n_0 \leq x \leq n_0^2$. By compactness there exist a_x, b_x such that $\tilde{S}(x) = S(a_x)S(b_x)$, $x = a_x b_x$, and $1 \leq a_x \leq b_x \leq n_0$. Then,

$$S(n_0) = \tilde{S}(n_0) \leq S(a_x)S(b_x - h/a_x)$$

(since $n_0 = a_x(b_x - h/a_x)$ and S is submultiplicative on $[1, n_0]$)

$$\begin{aligned} &\leq S(a_x)\Big(S(b_x) - \frac{ch}{a_x}\Big) \quad \text{(by (2.1))} \\ &= \tilde{S}(n_0 + h) - \frac{cS(a_x)}{a_x}h \\ &\leq \tilde{S}(n_0 + h) - (c/n_0)h, \end{aligned}$$

(since $S(a_x) \geq 1$ and $a_x \leq n_0$). So

$$S(n_0) + (c/n_0)h \leq \tilde{S}(n_0 + h).$$

Hence, provided $\varepsilon < c/n_0$, we have $S_\varepsilon(x) \leq \tilde{S}(x)$ for $1 \leq x \leq n_0^2$. To verify submultiplicativity of S_ε on the interval $[1, n_0^2]$ it remains to check that

$$S_\varepsilon(xy) \leq S_\varepsilon(x) S_\varepsilon(y)$$

for all $1 \leq x \leq n_o$ and $n_0 \leq y \leq n_0^2$ such that $xy \leq n_0^2$. Since $S_\varepsilon(xy) = S_\varepsilon(y) + \varepsilon(xy - y)$, we require

$$\frac{S_\varepsilon(y) + \varepsilon(x-1)y}{S_\varepsilon(y)} \leq S_\varepsilon(x),$$

i.e.,

$$\varepsilon(x-1)y \leq (S_\varepsilon(x) - 1)S_\varepsilon(y). \tag{2.2}$$

First consider the case $x \geq 2$. Then $S_\varepsilon(x) - 1 \geq 1$ and since $(x-1)y \leq xy \leq n_0^2$ it follows that (2.2) will be satisfied provided $n_0^2 \varepsilon \leq S(n_0)$. On the other hand, if $1 \leq x \leq 2$, then by condition (b) of Definition2.1, (2.2) reduces to $\varepsilon y \leq S_\varepsilon(y)$, which will again be satisfied provided $n_0^2 \varepsilon \leq S(n_0)$. This proves the lemma for

$$\varepsilon_0 = \min\left(\frac{c}{n_0}, \frac{S(n_0)}{n_0^2}\right). \qquad \blacksquare$$

By an obvious repeated application of Lemma 2.2 one obtains the following result.

LEMMA 2.3: *Suppose that $n_0 \geq 2$ and that S is submultiplicative on $[1, n_0]$. Then, given $\varepsilon > 0$ and $N_0 > n_0$, there exists a submultiplicative extension of S to $[1, N_0]$ such that $S(N_0) < S(n_0) + \varepsilon$.*

LEMMA 2.4: *Suppose that S is submultiplicative on $[1, n_0]$, where $n_0 \geq 2$, and that $S(n_0) = K \geq 2$. Then there exist $N_0 > n_0$ and a submultiplicative extension of S to $[1, N_0]$ such that $S(N_0) \geq 3K/2$.*

Proof: Let $n_1 = n_0^2$. By Lemma 2.2 we may and shall assume that S has been extended to be submultiplicative on $[1, n_1]$. By a second application of Lemma 2.2 there exists $\varepsilon > 0$ such that

$$S_\varepsilon(x) = \begin{cases} S(x) & \text{for } 1 \leq x \leq n_1 \\ S(n_1) + \varepsilon(x - n_1) & \text{for } n_1 < x \leq n_1^2 \end{cases}$$

is submultiplicative on $[1, 2n_1]$ (or even $[1, n_1^2]$ although we will only use submultiplicativity on $[1, 2n_1]$). If $S_\varepsilon(2n_1) \geq 3K/2$ then we are done. So we may assume that $S_\varepsilon(2n_1) < 3K/2$, which implies (since $S_\varepsilon(n_1) \geq K$) that

$$n_1\varepsilon < K/2. \tag{2.3}$$

Choose $N_0 > 2n_1$ such that $S_\varepsilon(N_0) = 3K/2$. We shall show that S_ε is submultiplicative on $[1, N_0]$. Suppose that $1 \leq x \leq y \leq xy \leq N_0$. Since $S_\varepsilon(x)$ is submultiplicative on $[1, 2n_1]$ we may assume that $xy \geq 2n_1$. Since $n_1 = n_0^2$, it follows that $y \geq n_0$, so $S_\varepsilon(y) \geq K$. First consider the case $x > 2$. Then $S_\varepsilon(x) \geq 2$, so

$$S_\varepsilon(xy) \leq S_\varepsilon(N_0) < 2K \leq S_\varepsilon(x)S_\varepsilon(y).$$

On the other hand, if $1 \leq x \leq 2$, then by condition (b) of Definition 2.1 $S_\varepsilon(x) = x$, so the submultiplicativity condition becomes

$$S_\varepsilon(y) + \varepsilon(x-1)y \leq S_\varepsilon(xy) \leq S_\varepsilon(x)S_\varepsilon(y) = xS_\varepsilon(y),$$

i.e., $\varepsilon y \leq S_\varepsilon(y)$. This is satisfied if $n_0 \leq y \leq K/\varepsilon$ since $S_\varepsilon(y) \geq S_\varepsilon(n_0) = K$. But

$$\begin{aligned} S_\varepsilon(K/\varepsilon) &= S_\varepsilon(n_1) + \varepsilon(K/\varepsilon - n_1) \geq K + K - n_1\varepsilon \geq K + K - K/2 \\ &= 3K/2 = S_\varepsilon(N_0), \end{aligned}$$

where the last inequality follows from (2.3). Thus, $K/\varepsilon \geq N_0$, which proves that S_ε is submultiplicative on $[1, N_0]$ as desired. ∎

By a repeated application of Lemma 2.4 one obtains the following result.

Lemma 2.5: *Suppose that $n_0 \geq 2$ and that S is submultiplicative on $[1, n_0]$. Then, given $M > 0$, there exist $N_0 > n_0$ and a submultiplicative extension of S to $[1, N_0]$ such that $S(N_0) > M$.*

Next we construct an infinite collection of mutually incomparable submultiplicative functions. This will be used to construct spreading model diagrams in Theorems 2.9 and 2.10 below. (In fact, the existence of arbitrarily large finite sets of incomparable submultiplicative functions would suffice for the applications.)

Proposition 2.6: *There exists a sequence $(S_i)_{i=1}^\infty$ of submultiplicative functions on $[1, \infty)$ such that for every non-empty finite set $A \subset \mathbb{N}$ and for every $j \in \mathbb{N} \setminus A$, we have*

$$\sup_{n \geq 1} \frac{S_j(n)}{\max_{i \in A} S_i(n)} = \infty. \tag{2.4}$$

Proof: We shall define $(S_i)_{i=1}^{\infty}$ on $[1,\infty)$ by defining their values inductively on an increasing sequence of initial segments $[1,n_0]$. Let us describe the inductive step. Suppose that (S_i) have been defined to be submultiplicative on some initial segment $[1,n_0]$ in such a way that the collection of restrictions of (S_i) to $[1,n_0]$ is a **finite** collection of functions on $[1,n_0]$. Now fix a finite set $A \subseteq \mathbb{N}$ and a positive integer N. By applying Lemma 2.3 to S_i $(i \in A)$ on $[1,n_0]$, and applying Lemma 2.5 to S_i $(i \in \mathbb{N}\setminus A)$ on $[1,n_0]$ (which is a finite collection by assumption), there exist $N_0 > n_0$ and submultiplicative extensions of S_i to $[1,N_0]$ such that

$$\max_{n\in[n_0,N_0]} \frac{S_j(n)}{\max_{i\in A} S_i(n)} > N$$

for all $j \in \mathbb{N}\setminus A$. At the end of the inductive step we have defined (S_i) to be submultiplicative on $[1,N_0]$. Moreover, the new collection of initial segments of $(S_i)_{i=1}^{\infty}$ on $[1,N_0]$ thus obtained will be finite. Now one simply enumerates (in any manner) the countable collection of possible choices for A and N to carry out the inductive definition. ∎

Let $1 \le p < \infty$ and let $w = (w(n))_{n=1}^{\infty}$ be a non-increasing sequence of positive **weights** such that $w(1) = 1$, $w(n) \to 0$ as $n \to \infty$, and $\sum_{n=1}^{\infty} w(n) = \infty$. Recall that the Lorentz sequence space $d(w,p)$ is the Banach space with Schauder basis (e_n) whose norm is defined by

$$\left\| \sum_{n=1}^{\infty} a_n e_n \right\|_{w,p} := \left(\sum_{n=1}^{\infty} a_n^{*p} w(n) \right)^{1/p},$$

where $(a_n^*)_{n=1}^{\infty}$ is the non-increasing rearrangement of any scalar sequence $(|a_n|)_{n=1}^{\infty}$ which converges to zero. Note that

$$\left\| \sum_{n=1}^{\infty} a_n e_n \right\|_{w,p} \le \left\| \sum_{n=1}^{\infty} a_n e_n \right\|_{p} := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}.$$

The corresponding **fundamental function** $(S(n))_{n=1}^{\infty}$ is defined by

$$S(n) = \left\| \sum_{i=1}^{n} e_i \right\|_{w,p}^{p} = \sum_{i=1}^{n} w(i).$$

It is known that $d(w,p)$ contains subspaces that are almost isometric to ℓ_p and is reflexive if and only if $1 < p < \infty$.

The weight w is said to be **submultiplicative** if there exists a constant C such that $S(mn) \le CS(m)S(n)$ for all $m,n \in \mathbb{N}$. We require the following theorem due to Altshuler, Casazza and Lin.

FACT 2.7 ([ACL]): Suppose that w is submultiplicative. Then every normalized block basis in $d(w,p)$ has a subsequence which is equivalent to the unit vector basis of ℓ_p or to the unit vector basis of $d(w,p)$.

The above theorem has the following immediate corollary.

COROLLARY 2.8: *Suppose that w is submultiplicative. Then every spreading model of $d(w,1)$ generated by a weakly null sequence is equivalent to the unit vector basis of $d(w,1)$. For $1 < p < \infty$, every spreading model of $d(w,p)$ generated by a weakly null sequence is equivalent to the unit vector basis of ℓ_p or to the unit vector basis of $d(w,p)$.*

Note that to each submultiplicative function S defined on $[1,\infty)$ corresponds a submultiplicative weight sequence $w(n) := S(n)-S(n-1)$ where $S(0)=0$ (with constant $C=1$) whose fundamental function is $(S(n))_{n=1}^{\infty}$. Let w_i $(1 \le i < \infty)$ be the weight sequences corresponding to the submultiplicative functions S_i constructed in Proposition 2.6. Note that $\lim_{n\to\infty} w_i(n) = 0$ and $\sum_n w_i(n) = \infty$ for each i.

Now we come to the main results of this section. For $n \in \mathbb{N}$, let $P(n)$ denote the power set of $\{1,\dots,n\}$ partially ordered by inclusion.

THEOREM 2.9: *For each $n \in \mathbb{N}$, let $X_n(1) := (\sum_{i=1}^{n} \oplus d(w_i,1))_\infty$. Then $SP_w(X_n(1))$ is order-isomorphic to $P(n) \setminus \{\emptyset\}$.*

Proof: Let $(f_j)_{j=1}^{\infty}$ be a normalized spreading model for $X_n(1)$ generated by a weakly null sequence. Then there exists a non-empty $A \subseteq \{1,\dots,n\}$ and normalized spreading models $(f_j^i)_{j=1}^{\infty}$ of $d(w_i,1)$ $(i \in A)$, generated by weakly null sequences in $d(w_i,1)$, such that

$$\Big\| \sum_{j=1}^{\infty} a_j f_j \Big\| \approx \max_{i\in A} \Big\| \sum_{j=1}^{\infty} a_j f_j^i \Big\|.$$

Thus, by the first part of Corollary 2.8,

$$\Big\| \sum_{j=1}^{\infty} a_j f_j \Big\| \approx \max_{i\in A} \Big\| \sum_{j=1}^{\infty} a_j e_j \Big\|_{w_i,1}. \tag{2.5}$$

Conversely, the right-hand side of (2.5) defines a normalized spreading model $SP(A)$ for every non-empty $A \subseteq \{1,\dots,n\}$. Note that

$$\Big\| \sum_{j=1}^{m} f_j \Big\| \approx \max_{i\in A} S_i(m) \quad (m \in \mathbb{N}).$$

Thus, by (2.4) of Proposition 2.6, we have

$$A \subset B \Leftrightarrow SP(A) < SP(B)$$

for all nonempty $A, B \subseteq \{1, \ldots, n\}$. ∎

In the reflexive case ($1 < p < \infty$) we have to add an extra node on the top.

THEOREM 2.10: *Let $1 < p < \infty$ and, for each $n \in \mathbb{N}$, let*

$$X_n(p) := \Big(\sum_{i=1}^{n} \oplus d(w_i, p)\Big)_\infty .$$

Then $SP_w(X_n(p))$ is order-isomorphic to $(P(n) \cup \{\{1, \ldots, n+1\}\}) \setminus \{\emptyset\}$.

Proof: The proof is essentially the same as before. However, from the second part of Corollary 2.8, we obtain an extra spreading model equivalent to the unit vector basis of ℓ_p which dominates every other spreading model. This spreading model corresponds to $\{1, \ldots, n+1\}$ under the order-isomorphism. ∎

3. Countable lattices with a minimum element

Recall that a **lattice** is a partially ordered set in which any two elements have both a least upper bound and a greatest lower bound. The following theorem is the main result of this section.

THEOREM 3.1: *Let L be a countable lattice with a minimum element not containing an infinite increasing sequence. Then there exists a reflexive space X_L such that $SP_w(X_L)$ is order-isomorphic to L.*

Remark 3.2: Recall that $(SP_w(X), \leq)$ is always a semi-lattice, i.e., every two elements have a least upper bound (Fact 1.1), and that when countable it does not contain any infinite increasing sequences (Fact 1.3). It is easy to see that such a semi-lattice with a minimum element is automatically a lattice. Thus, Theorem 3.1 characterizes the possible poset structure of $(SP_w(X), \leq)$ when $SP_w(X)$ is countable and has a minimum element.

The space X_L will be an ℓ_p direct-sum of suitably constructed Orlicz sequence spaces. The proof of the theorem will be given at the end of the section. First we recall some preliminary facts about Orlicz spaces. All the unexplained terms and facts can be found in Chapter 4 of [LT], with which our notation is consistent.

An **Orlicz function** M is a real-valued continuous non-decreasing and convex function defined on $[0,1]$ such that $M(0)=0$ and $M(1)=1$. For a given M, the **Orlicz sequence space** ℓ_M is the space of all sequences of scalars $x=(a_1,a_2,\ldots)$ such that $\sum_{n=1}^{\infty} M(|a_n|/\rho)<\infty$ for some $\rho>0$, equipped with the norm

$$\|x\| = \inf\Big\{\rho>0 : \sum_{n=1}^{\infty} M(|a_n|/\rho)\le 1\Big\}.$$

We assume that M satisfies the Δ_2**-condition** at zero (i.e., that there exists $C>0$ such that $M(2t)\le CM(t)$ for all $0\le t\le 1/2$). Then the unit vectors form a normalized symmetric basis for ℓ_M. If N also satisfies the Δ_2-condition at zero, then M and N are **equivalent** if there exists a constant $C>0$ such that $(1/C)N(t)\le M(t)\le CN(t)$ for all $0\le t\le 1$.

If $C\ge 1$ and M and N are two Orlicz functions such that $N(t)\le CM(t)$ for all $0<t\le 1$, then the unit vector basis of ℓ_M C-dominates that of ℓ_N. Conversely, if M and N satisfy the Δ_2-condition at zero and the unit vector basis of ℓ_M dominates that of ℓ_N then there exists $C\ge 1$ such that $N(t)\le CM(t)$ for all $0<t\le 1$.

If M satisfies the Δ_2-condition at zero then an Orlicz sequence space ℓ_N is isomorphic to a subspace of ℓ_M if and only if N is equivalent to some function in $C_{M,1}$, where $C_{M,1}$ is the norm-closed convex hull in $C[0,1]$ of the set

$$E_{M,1} = \overline{\Big\{\frac{M(\lambda t)}{M(\lambda)};0<\lambda<1\Big\}}. \tag{3.1}$$

See [LT, Lemma 4.a.6, Remark (p. 141) and Theorem 4.a.8] for this result.

As noted in [S1], this is easily generalized to the spreading models of ℓ_M: $(\tilde{x_i})$ is a spreading model generated by a normalized block sequence (x_i) in ℓ_M if and only if $(\tilde{x_i})$ is equivalent to the unit vector basis of ℓ_N for some $N\in C_{M,1}$. Moreover, it follows from the proofs of [LT, Theorem 4.a.8] and [S1, Proposition 2.2] that if M satisfies Δ_2-condition and $N\in C_M:=\bigcap_{0<\Lambda} C_{M,\Lambda}$ (see [LT, Lemma 4.a.6]), then one gets the equivalence **isometrically**.

We will use the following method of representing Orlicz functions by sequences of zeros and ones, introduced by Lindenstrauss and Tzafriri [LT, p. 161].

Fix $0<\tau<1$ and $1<r<p<\infty$. For every sequence of zeros and ones, $\eta=(\eta(n))_{n=1}^{\infty}$ (i.e. $\eta(n)\in\{0,1\}$ for all n), let M_η be the piecewise linear function defined on $[0,1]$ satisfying $M_\eta(0)=0$, $M_\eta(1)=1$, and

$$M_\eta(\tau^k)=\tau^{rk+(p-r)\sum_{n=1}^{k}\eta(n)},\quad k=1,2,\ldots.$$

LEMMA 3.3: *Fix $0<\tau<1$ and $1<r<p<\infty$. If $p-r$ is sufficiently small, then, for all η, M_η is an Orlicz function satisfying the Δ_2-condition at zero.*

Proof: To show that M_η is convex it suffices to check that the slope of the chord joining $(\tau^{n+1}, M_\eta(\tau^{n+1}))$ to $(\tau^n, M_\eta(\tau^n))$ is a decreasing function of n, i.e.

$$\frac{M_\eta(\tau^n)-M_\eta(\tau^{n+1})}{\tau^n-\tau^{n+1}} \le \frac{M_\eta(\tau^{n-1})-M_\eta(\tau^n)}{\tau^{n-1}-\tau^n}. \tag{3.2}$$

Using the fact that $M_\eta(\tau^{k+1}) = \tau^r M_\eta(\tau^k)$ if $\eta(k+1)=0$ and $M_\eta(\tau^{k+1}) = \tau^p M_\eta(\tau^k)$ if $\eta(k+1)=1$, (3.2) simplifies to the following pair of conditions:

$$\tau^{r-1}(1-\tau^p) \le 1-\tau^r \quad \text{and} \quad \tau^{p-1}(1-\tau^r) \le 1-\tau^p. \tag{3.3}$$

Both conditions are clearly satisfied if $p-r$ is sufficiently small. (Note that the first condition is *not* satisfied, however, if r is very close to 1.) The Δ_2-condition is easily checked. ∎

Henceforth, we shall always assume that τ, p and r satisfy the conclusion of Lemma 3.3.

PROPOSITION 3.4: *Suppose that $1<p<2$. Then there exists $C<\infty$ such that for each sequence η of zeros and ones, ℓ_{M_η} C-embeds into $L_1[0,1]$.*

Proof: Observe that the inequalities in (3.3) are reversed if $0<r<p<1$. This implies that $M_\eta(\sqrt{t})$ is equivalent to a *concave* function if $p<2$. By a result of Bretagnolle and Dacuhna-Castelle [BD] ℓ_{M_η} embeds isomorphically into $L_1[0,1]$. To see that there exists a uniform embedding constant, observe that there exists a 'universal' sequence $\rho=(\rho(n))_{n=1}^\infty$ such that every sequence of zeros and ones, η, is a pointwise limit of the collection $\{(\rho(n+k)_{n=1}^\infty \colon k\in\mathbb{N}\}$ of left shifts of ρ: indeed, let ρ be the concatenation of all possible finite sequences of zeros and ones. It follows that $E_{M_\rho} := \bigcap_{\Lambda>0} E_{M_\rho,\Lambda} \subseteq C_{M_\rho}$ (see (3.1)) contains M_η for every η, and hence that ℓ_{M_η} is isometric to a spreading model of ℓ_{M_ρ} for every η. Finally, since ℓ_{M_ρ} C-embeds into $L_1[0,1]$ for some $C<\infty$, it follows that ℓ_{M_η} C-embeds into $L_1[0,1]$ for every η. ∎

We will be interested only in a simple class of such spaces as described in the following.

LEMMA 3.5: *Let $1<r<p<\infty$, and $0<\tau<1$. Let $(n_k)\subset\mathbb{N}$ satisfy $n_1=1$ and $n_{k+1}-n_k\uparrow\infty$, and put*

$$\rho(i)=\begin{cases}0 & \textit{if } i=n_k\\ 1 & \textit{otherwise.}\end{cases}$$

Let $M := M_\rho$ be the corresponding Orlicz function. Then ℓ_M satisfies the following:

(a) *Every spreading model is τ^{-2p}-dominated by the unit vector basis of ℓ_M.*
(b) *Every spreading model is equivalent either to the unit vector basis of ℓ_M or to the unit vector basis of ℓ_p.*
(c) *Every spreading model that is equivalent to the unit vector basis of ℓ_p is actually τ^{-5p}-equivalent to the unit vector basis of ℓ_p.*
(d) *ℓ_M is reflexive.*

Proof: Observe that

$$\sum_{i=1}^{n} \rho(i) \leq \sum_{i=k+1}^{k+n} \rho(i) \quad \text{for all } k, n \in \mathbb{N}.$$

Therefore, for all $\lambda = \tau^k$ and $t = \tau^n$, we have

$$\begin{aligned}\frac{M(\lambda t)}{M(\lambda)} &= \frac{\tau^{r(k+n)+(p-r)\sum_{i=1}^{k+n}\rho(i)}}{\tau^{rk+(p-r)\sum_{i=1}^{k}\rho(i)}} = \tau^{rn+(p-r)\sum_{i=k+1}^{k+n}\rho(i)} \\ &\leq \tau^{rn+(p-r)\sum_{i=1}^{n}\rho(i)} = M(t).\end{aligned}$$

A simple calculation now yields, that

$$\frac{M(\lambda t)}{M(\lambda)} \leq \tau^{-2p} M(t), \tag{3.4}$$

for all $0 < \lambda, t < 1$.

Now let $N \in C_{M,1}$. Then, by the definition of $C_{M,1}$, N is the limit in the uniform norm of a sequence of convex combinations (F_n) of the form

$$F_n = \sum_{i \in A_n} a_i \frac{M(\lambda_i t)}{M(\lambda_i)},$$

for some finite $A_n \subset \mathbb{N}$, $0 < \lambda_i \leq 1$, and positive (a_i) with $\sum_{i \in A_n} a_i = 1$.

Thus (3.4) implies that for all $N \in C_{M,1}$,

$$N(t) \leq \tau^{-2p} M(t) \quad \text{for all } 0 < t < 1, \tag{3.5}$$

which proves (a). To see that N is equivalent either to M or to t^p, we distinguish between two cases corresponding to the manner in which the sequence (F_n) converges to N.

For the first case, suppose that there exist $n_0 \in \mathbb{N}$, $\bar{\lambda} > 0$ and $\delta > 0$ such that for all $n \geq n_0$,

$$\sum_{\lambda_i \geq \bar{\lambda}, i \in A_n} a_i \geq \delta.$$

It follows that

$$N(t) \geq \delta M(\bar{\lambda} t), \quad \text{for all } t > 0,$$

which, along with (3.5), implies that N is equivalent to M.

For the second case, we suppose that for all $n_0 \in \mathbb{N}$, $\bar{\lambda} > 0$ and $\delta > 0$, there exists $n \geq n_0$ such that

$$\sum_{\lambda_i \geq \bar{\lambda}, i \in A_n} a_i < \delta.$$

Fix $t = \tau^m$. Since $n_{k+1} - n_k \uparrow \infty$, it follows that every m consecutive terms of ρ which begin sufficiently far along the sequence can contain at most one zero term. This implies that if $\lambda = \tau^k$ is sufficiently small then

$$t^p \leq \frac{M(\lambda t)}{M(\lambda)} \leq \tau^{-p} t^p. \tag{3.6}$$

Now fix $\delta > 0$ and $0 < t < 1$. It follows easily from (3.6) that there exists $\bar{\lambda} > 0$ such that for all $\lambda < \bar{\lambda}$

$$\tau^{2p} t^p \leq \frac{M(\lambda t)}{M(\lambda)} \leq \tau^{-3p} t^p. \tag{3.7}$$

By assumption there exists $n \in \mathbb{N}$ such that

$$|F_n(t) - N(t)| < \delta \quad \text{and} \quad \sum_{\lambda_i \geq \bar{\lambda}, i \in A_n} a_i < \delta. \tag{3.8}$$

Now (3.7) gives

$$F_n(t) = \sum_{\lambda_i < \bar{\lambda}, i \in A_n} a_i \frac{M(\lambda_i t)}{M(\lambda_i)} + \sum_{\lambda_i \geq \bar{\lambda}, i \in A_n} a_i \frac{M(\lambda_i t)}{M(\lambda_i)} \leq \tau^{-3p} t^p + \delta.$$

A similar calculation yields

$$F_n(t) \geq (1 - \delta) \tau^{2p} t^p - \delta.$$

Since $|N(t) - F_n(t)| < \delta$ and $\delta > 0$ is arbitrary, we get

$$\tau^{2p} t^p \leq N(t) \leq \tau^{-3p} t^p.$$

This proves (c) and also completes the proof of (b). Finally, (b) and [LT, Lemma 4.a.6] imply that ℓ_1 is not isomorphic to a subspace of ℓ_M, which in turn implies by [LT, Proposition 4.a.4] that ℓ_M is reflexive, which proves (d). ∎

We will also make use of the following general fact.

LEMMA 3.6: *Let $X = (\sum_{j=1}^{\infty} \oplus X_j)_p$, where $1 \leq p < \infty$ and each X_j is an infinite-dimensional Banach space with $SP_w(X_j) \neq \emptyset$ for all j, and let $(\tilde{x}_i) \in SP_w(X)$ be a spreading model generated by a normalized weakly null sequence in X. Then there exist non-negative $(c_j)_{j=0}^{\infty}$ with $\sum_{j=0}^{\infty} c_j^p = 1$ and normalized spreading models $(\tilde{x}_i^j)_i \in SP_w(X_j)$ such that for all scalars (a_i)*

$$\Big\| \sum_i a_i \tilde{x}_i \Big\| = \Big[\sum_{j=1}^{\infty} c_j^p \Big\| \sum_i a_i \tilde{x}_i^j \Big\|^p + c_0^p \sum_i |a_i|^p \Big]^{1/p}.$$

Proof: Suppose that the normalized weakly null sequence (y_i) generates the spreading model $(\tilde{x}_i)$. Write $y_i = (y_i^j)_{j=1}^{\infty}$, where $y_i^j \in X_j$ for each j. By a diagonalization argument, (y_i) has a subsequence (x_i) such that

$$\lim_{i\to\infty} \|x_i^j\| = c_j \quad \text{and} \quad \sup_{i\geq j} |\|x_i^j\| - c_j| \leq \frac{1}{2^j}, \tag{3.9}$$

and $(c_j^{-1} x_i^j)_{i=1}^{\infty}$ generates a normalized spreading model $(\tilde{x}_i^j)_i \in SP_w(X_j)$. Note that (3.9) implies that $c_0 := \lim_{i\to\infty} \|x_i - P_i(x_i)\|$ exists, where $P_i(x_i) = (x_i^1, x_i^2, \ldots, x_i^i, 0, 0, 0, \ldots)$. One now checks that the spreading model $(\tilde{x}_i)$ generated by (x_i) is given by the stated formula. (Note also that $\sum_{j=0}^{\infty} c_j^p = 1$ since (x_i) is normalized.) ∎

Before proving Theorem 3.1 we give an application of Lemma 3.6.

THEOREM 3.7: *Let X be an infinite-dimensional Banach space such that $SP_w(X)$ is a countable chain. Then there exists a countable ordinal α such that $SP_w(X)$ is order-isomorphic to α with the* **reverse order**. *Conversely, if $\alpha \geq 1$ is a countable ordinal then there exists a reflexive Banach space X such that $SP_w(X)$ is order-isomorphic to α with the reverse order.*

Proof: For the first part, by Fact 1.3 $SP_w(X)$ does not admit an infinite strictly increasing sequence. Thus the reverse order on $SP_w(X)$ is a well-ordering and hence is order-isomorphic to a countable ordinal. For the converse, let $\beta \mapsto p_\beta$ $(\beta < \alpha)$ be an increasing order-isomorphism from α onto a subset of $[2, 3]$ such that $p_0 = 2$, and set $X := (\sum_{\beta<\alpha} \oplus \ell_{p_\beta})_2$. Using the well-foundedness of α and the monotonicity property of the ℓ_p norms (i.e., that $\|\cdot\|_q \leq \|\cdot\|_p$ if $p \leq q$), it follows easily from Lemma 3.6 that every normalized spreading model of X is equivalent to the unit vector basis of ℓ_{p_β} for some $\beta < \alpha$; so $SP_w(X)$ is order-isomorphic to α with the reverse order. ∎

We now proceed to the

Proof of Theorem 3.1: For convenience we shall assume that L is countably infinite. (When L is finite only minor notational changes are needed.) The space X_L will be of the form $X_L = \left(\sum_{j=0}^{\infty} \oplus \ell_{M_j}\right)_p$ for suitably constructed Orlicz sequence space ℓ_{M_j}'s, with $M_j := M_{\rho_j}$ for certain sequences ρ_j of zeros and ones (for the same τ, r and p). The 'patterns' of the ρ_j's will be of the form

$$\rho_j(i) = \begin{cases} 0 & \text{if } i \in \sigma(j) \\ 1 & \text{otherwise.} \end{cases}$$

for some fast increasing sequence $\sigma(j) \subset \mathbb{N}$, with $1 \in \sigma(j)$. For simplicity, for every j we will take $\sigma(j)$ to be a subset of $\mathcal{M} = \{1, 2, 2^2, 2^3, \ldots\}$ which will ensure that the hypothesis of Lemma 3.5 is satisfied.

The patterns of the ρ_j's (equivalently, the σ_j's) will be developed inductively on finite intervals of $\mathbb{N}$ according to a two-step procedure which we call **(ε, A)-domination**.

Let $A \subset \mathbb{N}$ and $\varepsilon > 0$. Suppose that for some $N \in \mathbb{N}$, the ρ_j's have already been defined on the initial segment $[1, N]$ so that

$$\sum_{i=1}^{N} \rho_j(i) = \sum_{i=1}^{N} \rho_k(i), \quad \text{for all } j, k \in \mathbb{N}. \tag{3.10}$$

The (ε, A)-domination procedure extends the definition of the ρ_j's to an intial segment $[1, N_1]$ for some $N_1 > N$. Let us first dispose of some trivial cases. If $A = \emptyset$ or if $A = \mathbb{N}$ then set $N_1 = N + 1$ and $\rho_j(N_1) = 1$ for all j.

Now suppose that both A and $\mathbb{N} \setminus A$ are non-empty. The first step of the procedure is carried out as follows. Choose a sufficiently large (just how large is specified below) integer $m > N$. For all $k \in \mathbb{N} \setminus A$ place 0's on the coordinates from $[N+1, m] \cap \mathcal{M}$ of the ρ_k's (while the rest of the coordinates of the interval are filled with 1's), and for all $j \in A$ place 1's on **all** the coordinates from $[N+1, m]$ of the ρ_j's, where m is chosen so that

$$\sum_{i=1}^{m} \rho_j(i) - \sum_{i=1}^{m} \rho_k(i)$$

is sufficiently large to ensure that

$$\frac{M_{\rho_j}(\tau^m)}{M_{\rho_k}(\tau^m)} < \varepsilon, \quad \text{for all } j \in A,\ k \in \mathbb{N} \setminus A.$$

For the second step we choose a sufficiently large integer $N_1 > m$ (just how large is specified below), with $N_1 \in \mathcal{M}$, and 'rebalance' all of the ρ_j's on the interval $[m+1, N_1]$. This is achieved by placing 0's on the coordinates from $[m+1, N_1] \cap \mathcal{M}$ for all the ρ_j's ($j \in A$) and by placing 1's on the coordinates from $[m+1, N_1]$ for all the ρ_k's ($k \in \mathbb{N} \setminus A$), where N_1 is chosen so that (3.10) is satisfied with N replaced by N_1. At the end of this second step the M_j's are equal again, i.e.

$$M_j(\tau^{N_1}) = M_k(\tau^{N_1}) \quad \text{for all } j, k \in \mathbb{N}.$$

We now pass to the main construction. Let $L = \{e_0, e_1, e_2, \dots, \}$ be the given countable lattice, where e_0 is the minimum element. Consider $\bar{L} = \{\bar{e}_1, \bar{e}_2, \dots\}$, where $\bar{e}_j = \{i \in \mathbb{N} : e_i \leq e_j\}$ for all $j \in \mathbb{N}$. Put $\rho_0 = (1, 1, 1, \dots)$.

We begin by setting $\rho_j(1) = 0$ for all $j \in \mathbb{N}$, which ensures that the ρ_j's satisfy Lemma 3.5. Now, for every $j \in \mathbb{N}$ and every $\varepsilon = 2^{-k}$, $k = 1, 2, \dots$, we carry out an (ε, A)-domination procedure for $A = \bar{e}_j$. After enumerating the collection of all such pairs (ε, A), which is countable, we carry out all the (ε, A)-dominations in the order specified by this enumeration.

The resulting sequences $\rho_0, \rho_1, \rho_2, \dots$ have the following properties.

(i) M_{ρ_0} is equivalent to the function t^p.

(ii) For all $i, j \in \mathbb{N} \cup \{0\}$, there exists a constant $C < \infty$ such that

$$M_{\rho_i}(t) \leq C M_{\rho_j}(t) \quad \text{for all } 0 < t < 1$$

if and only if $e_i \leq e_j$ in $(L, \leq)$. Moreover, if there exists such a C then we can choose $C = 1$.

(iii) For every non-empty finite set $F \subset \mathbb{N} \cup \{0\}$

$$\max_{j \in F} M_{\rho_j} = M_{\rho_{j_0}}, \quad \text{where } e_{j_0} = \bigvee_{j \in F} e_j.$$

Proof of (iii): To derive a contradiction, assume that there exists $t = \tau^m$ such that $\max_{j \in F} M_{\rho_j}(t) < M_{\rho_{j_0}}(t)$. Because of the 'rebalancing' step in the domination procedure, it follows that m belongs to an interval of $\mathbb{N}$ where an (ε, A)-domination takes place for some A such that $F \subseteq A$ and $j_0 \in \mathbb{N} \setminus A$. There exists $k \in \mathbb{N}$ such that $A = \bar{e}_k$. Then $e_j \leq e_k$ for all $j \in F$. Since L is a lattice it follows that $e_{j_0} \leq e_k$, and hence $j_0 \in A$, which is the desired contradiction.

(iv) Let B be a non-empty subset of $\mathbb{N} \cup \{0\}$ Then there exists a finite subset F of B such that

$$\max_{j \in B} M_{\rho_j} = \max_{j \in F} M_{\rho_j}.$$

Proof of (iv): Suppose not. Then there exists $(j_k)_{k=1}^{\infty} \subset B$ such that for all $n \in \mathbb{N}$

$$\max_{1 \leq k \leq n} M_{\rho_{j_k}} < \max_{1 \leq k \leq n+1} M_{\rho_{j_k}}.$$

This, however, implies by (iii) and (ii) that

$$\bigvee_{1\le k\le n} e_{j_k} < \bigvee_{1\le j\le n+1} e_{j_k}, \quad \text{for each } n.$$

But this contradicts our assumption that there are no increasing infinite sequences in L.

Now consider

$$X_L = \Big(\sum_{j=0}^{\infty} \oplus \ell_{M_j}\Big)_p,$$

where $M_j = M_{\rho_j}$, $j \in \mathbb{N}\cup\{0\}$. By (d) of Lemma 3.5 each ℓ_{M_j} is reflexive and hence X_L is also reflexive. It follows from property (ii) that the *collection* of spreading models generated by the unit vector bases of each ℓ_{M_j} is order-isomorphic to L. Therefore it remains to show that *every* spreading model of X_L is equivalent to the unit vector basis $(b_i^j)_i$ of ℓ_{M_j} for some $j \in \mathbb{N}\cup\{0\}$.

Let $(\tilde{x}^i)$ be a normalized spreading model of X_L. Then, for all $(a_i) \in c_{00}$, we have by Lemma 3.6

$$\Big\|\sum_i a_i\tilde{x}_i\Big\| = \Big[\sum_j c_j^p \Big\|\sum_i a_i\tilde{x}_i^j\Big\|^p + c_0^p\sum_i |a_i|^p\Big]^{1/p}, \tag{3.11}$$

where $(\tilde{x}_i^j)_i$ is a normalized spreading model of ℓ_{M_j} and $(c_j)_{j=0}^{\infty}$ belongs to the non-negative unit sphere of ℓ_p.

Let B be the collection of all $j \in \mathbb{N}$ such that $c_j \neq 0$ and such that $(\tilde{x}_i^j)_i$ is equivalent to $(b_i^j)_i$. If $j \notin B$ then either $c_j = 0$ or, by Lemma 3.5, $(\tilde{x}_i^j)_i$ is τ^{-5p}-equivalent to the unit vector basis of ℓ_p. Thus, if $B = \emptyset$, then (3.11) implies that $(\tilde{x}^i)$ is equivalent to the unit vector basis of ℓ_p and hence to $(b_i^0)_i$. So suppose that $B \neq \emptyset$; then, by Lemma 3.5, each $(\tilde{x}_i^j)_i$ $(j \in B)$ is τ^{-2p}-dominated by $(b_i^j)_i$. By properties (iii) and (iv) above there exist a finite set $F \subset B$ and $j_0 \in \mathbb{N}$ such that

$$\max_{j\in B} M_{\rho_j} = \max_{j\in F} M_{\rho_j} = M_{\rho_{j_0}}.$$

Hence there exists $0 \le K < \infty$ such that

$$\begin{aligned}\Big\|\sum_i a_i\tilde{x}_i\Big\| &\le \Big[\tau^{-2p}\sum_{j\in B} c_j^p \Big\|\sum_i a_i b_i^j\Big\|^p + K\sum_i |a_i|^p\Big]^{1/p}\\ &\le \Big[\big(\tau^{-2p}\sum_{j\in B} c_j^p\big)\Big\|\sum_i a_i b_i^{j_0}\Big\|^p + K\sum_i |a_i|^p\Big]^{1/p},\end{aligned}$$

which implies that $(\tilde{x}_i)_i \le (b_i^{j_0})$. On the other hand, there exists $c > 0$ such that

$$\Big\|\sum_i a_i \tilde{x}_i\Big\| \ge \Big[\sum_{j\in F} c_j^p \Big\|\sum_i a_i \tilde{x}_i^j\Big\|^p\Big]^{1/p} \ge c \max_{j\in F}\Big\|\sum_i a_i b_i^j\Big\|$$

(since $(\tilde{x}_i^j)_i$ is equivalent to $(b_i^j)_i$ for each $j \in F$)

$$\ge \frac{c}{\operatorname{card} F}\Big\|\sum_i a_i b_i^{j_0}\Big\|.$$

Thus, $(\tilde{x}_i)_i$ is equivalent to $(b_i^{j_0})_i$. ∎

Remark 3.8: For each $1 < p < \infty$, the above construction yields the unit vector basis of ℓ_p as the minimum element of $SP_w(X_L)$. If we allow X_L to be non-reflexive we can obtain c_0 as the minimum element. However, this requires a rather different construction. Using results of Casazza and Lin [CL], it is possible to construct a c_0-sum of duals of certain Lorentz sequence spaces for which c_0 is the minimum element of $SP_w(X_L)$. We omit the details of this result.

Remark 3.9: Let ρ be the universal sequence used in the proof of Proposition 3.4. It follows from the proof of Theorem 3.1 that $SP_w(\ell_{M_\rho})$ contains a **subset** that is order-isomorphic to any given countable poset P. (Note also that there is a universal countable poset.) By Proposition 3.4, ℓ_{M_ρ} is isomorphic to a reflexive subspace of $L_1[0,1]$ when $p < 2$.

Corollary 3.10: *For every finite lattice L there exists a reflexive space X_L such that $SP_w(X_L)$ is order-isomorphic to L.*

Corollary 3.11: *Let L be a finite lattice (resp. countable lattice with a minimum element and without any infinite increasing sequence). Then there exists a reflexive (resp. non-reflexive) subspace Y_L of $L_1[0,1]$ such that $SP_w(Y_L)$ is order-isomorphic to L.*

Proof: Using the notation of Theorem 3.1, let

$$Y_L = \Big(\sum_{j=0}^{\infty} \oplus \ell_{M_j}\Big)_1 .$$

By Proposition 3.4, if $p < 2$ then for some $C < \infty$ each ℓ_{M_j} C-embeds into $L_1[0,1]$, and hence Y_L is isomorphic to a subspace of $L_1[0,1]$. Moreover, Y_L

is reflexive if and only if L is finite. The proof of Theorem 3.1 shows that $SP_w(Y_L)$ is order-isomorphic to L. (Note that if L is infinite then Y_L also has an ℓ_1 spreading model that is **not** generated by a weakly null sequence.) ∎

Added in proof. Recently, P. Dodos has shown that for a separable space X if $SP_w(X)$ is uncountable, then it contains an antichain of the size of the continuum. This answers Problem 1.13.

Moreover, D. Leung and W. K. Tang have informed us that they have answered Problem 1.15 affirmatively in non-reflexive case, using a Lorentz space construction similar to the one developed in Section 2.

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