ON THE COMPLEXITY OF THE UNIFORM HOMEOMORPHISM RELATION BETWEEN SEPARABLE BANACH SPACES

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Abstract. In this paper we investigate the uniform homeomorphism relation between separable Banach spaces and the related relation of local equivalence. We completely characterize the descriptive complexity of local equivalence in the Borel reducibility hierarchy. This also provides a lower bound for the complexity of the uniform homeomorphism.

Contents

1. Introduction

Recently, there has been a growing interest in understanding the complexity of common analytic equivalence relations between separable Banach spaces via the notion of Borel reducibility in descriptive set theory (see [Bos] [FG] [FLR] [FR1] [FR2] [Me]). In general, the notion of Borel reducibility yields a hierarchy (though not linear) among equivalence relations in terms of their relative complexity.

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The most important relations between separable Banach spaces include the isometry, the isomorphism, the equivalence of bases, and in nonlinear theory, Lipschitz and uniform homeomorphisms. The exact complexity of the first four relations has been completely determined by recent work in the field. Using earlier work of Weaver [W], Melleray [Me] proved that the isometry between separable Banach spaces is a universal orbit equivalence relation. Rosendal [Ro] studied the equivalence of bases and showed that it is a complete K_σ equivalence relation. Using the work of Argyros and Dodos [AD] on amalgamations of Banach spaces, Ferenczi, Louveau, and Rosendal [FLR] recently showed that the isomorphism, the (complemented) biembeddability and the Lipschitz equivalence between separable Banach spaces, as well as the permutative equivalence of Schauder bases, are complete analytic equivalence relations. The Borel reducibility among these equivalence relations, as well as some other equivalence relations we will be dealing with in this paper, is illustrated in Figure 1 (see Section 2 below for the definitions of the equivalence relations). Note that in particular the complete analytic equivalence relation $E_{\mathbf{\Sigma}^1_1}$ is the most complex one in the Borel reducibility hierarchy of all analytic equivalence relations.

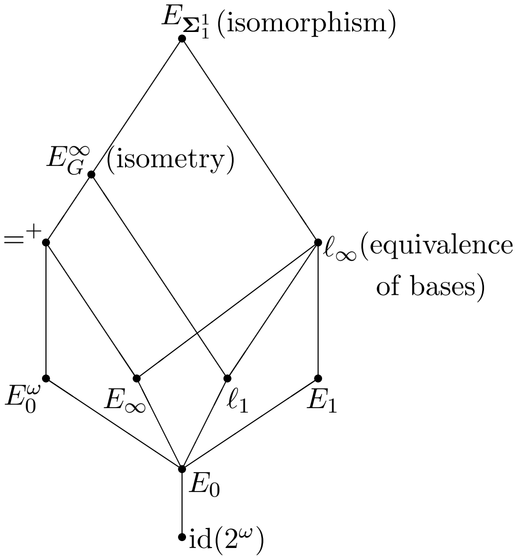

FIGURE 1. Equivalence relations characterizing the complexity of classification problems for Banach spaces.

The main problem left open now is to determine the exact complexity of the uniform homeomorphism between separable Banach spaces (see Problem 23, [FLR]). Recall that Banach spaces X and Y are uniformly homeomorphic if there exists a uniformly continuous bijection $\phi : X \to Y$ such that ϕ^{-1} is also uniformly continuous. The understanding of the uniform homeomorphism relation between Banach spaces, also known as the uniform classification of Banach spaces, is in fact one of the main programs in the nonlinear theory of Banach spaces.

Compared to the linear theory, results about the uniform classification are significantly harder to prove, and their proofs often use a combination of metric, geometric, and topological arguments (for a good survey of methods and results see Chapter 10, [BL]). Moreover, previous efforts have been mostly focused on the uniform classification of classical Banach spaces. For instance, it is well-known that for $1 \leq p \neq q < \infty$, ℓ_p and ℓ_q are not uniformly homeomorphic (due to Ribe [Ri]). Also, for $p \neq 2$ ℓ_p and L_p are not uniformly homeomorphic (due to Bourgain [Bou] for $1 \leq p < 2$ and Gorelik [Go] for $2 < p < \infty$). In fact, it turns out that for $1 < p < \infty$, the uniform structure of ℓ_p completely determines its linear structure (a result due to Johnson, Lindenstrauss, and Schechtman [JLS]). This also generalizes to certain finite sums of ℓ_p spaces.

From the point of view of descriptive set theory all previously known results on the uniform classification give some lower bound estimates for its complexity. However, these lower bounds are no more complex than $\mathrm{id}(2^\omega)$, the least complicated one in Figure 1. In contrast to this, it is conceivable that the uniform classification is as complex as Lipschitz and isomorphic classifications, that is, it is $E_{\mathbf{\Sigma}^1_1}$. Thus there is a huge gap between what was conjectured and what could be verified.

In this paper, we give a slightly improved lower bound for the complexity of the uniform classification. We show that the complete K_σ equivalence relation (represented by the equivalence relation ℓ_∞, to be defined in Section 2) is Borel reducible to the uniform homeomorphism relation between separable Banach spaces. As shown in Figure 1, this in particular implies that the uniform homeomorphism relation is at least as complex as the equivalence of bases.

The study of the uniform classification is essentially devoted to understanding what aspects of the linear structure of Banach spaces are invariant under uniform homeomorphisms. As an important example, a fundamental theorem of Ribe [Ri] asserts that the local structure of finite-dimensional subspaces is such an invariant. The proof of our main theorem is a straightforward application of this theorem of Ribe. Moreover, we will isolate a concept of local equivalence between separable Banach spaces in Section 3 and prove in Section 5 that it is Borel bireducible with ℓ_∞. This means that the lower bound we have reached for the complexity of the uniform classification is the best possible with the consideration of local structures.

We can now state the main theorems of this paper.

Theorem 1.1. *There exists a Borel family* $\{S_{\vec{x}} : \vec{x} \in \mathbb{R}^\omega\}$ *of separable Banach spaces such that the following are equivalent for any* $\vec{x}, \vec{y} \in \mathbb{R}^\omega$*:*

(a) $\vec{x} - \vec{y} \in \ell_\infty$*;*
(b) $S_{\vec{x}}$ *and* $S_{\vec{y}}$ *are uniformly homeomorphic;*
(c) $S_{\vec{x}}$ *and* $S_{\vec{y}}$ *are isomorphic;*

(d) *$S_{\vec{x}}$ and $S_{\vec{y}}$ are locally equivalent.*

Theorem 1.2. *The local equivalence between separable Banach spaces is Borel bireducible with ℓ_∞.*

Of course in general the local structure is not sufficient to determine the uniform structure (for instance, as the results of Bourgain and Gorelik mentioned above show). It is anticipated that the complexity of the uniform classification is much more complex than ℓ_∞. To verify this it would be enough to show that the equivalence relation E_0^ω is Borel reducible to the uniform homeomorphism relation. As Figure 1 suggests, E_0^ω is in some sense the least complex equivalence relation not Borel reducible to ℓ_∞.

In Section 6 we generalize the construction in the proof of Theorem 1.1 and consider a variety of classes of separable Banach spaces with a similar construction scheme. The uniform homeomorphism relations for these classes are all no more complex than ℓ_∞. We then determine exactly what kind of complexity the uniform homeomorphism relations on these classes can achieve. It turns out that they can only be ℓ_∞, E_1, E_0, or smooth.

Our constructions in Sections 4 and 6 will yield only classes of separable Banach spaces for which the uniform homeomorphism and the isomorphism relations coincide. In general it is well-known that the uniform homeomorphism is a genuinely coarser equivalence relation than the isomorphism (see, for example, Section 10.4, [BL]). Therefore it is of interest to study the question how many different isomorphism classes a single uniform homeomorphic class can contain. In Section 7 we prove the following related result.

Theorem 1.3. *There exists a Borel class $\mathcal{C}$ of mutually uniformly homeomorphic separable Banach spaces such that the equality relation of countable sets of real numbers, denoted $=^+$, is Borel reducible to the isomorphism relation on $\mathcal{C}$.*

The rest of the paper is organized as follows. In Section 2 we define all benchmark equivalence relations relevant to our discussions in this paper and review the Borel reducibility theory of equivalence relations on Polish spaces. In Section 3 we explain how to apply the framework of the descriptive set theory of equivalence relations to the uniform classification of separable Banach spaces. We also define the notion of local equivalence and show that it is $\boldsymbol{\Sigma}_3^0$ in two different codings of separable Banach spaces. In Section 4 we give the construction of the family in Theorem 1.1 and prove some basic properties. In Section 5 we give the proofs of both Theorems 1.1 and 1.2. We also generalize the ℓ_∞ equivalence relation and prove a dichotomy theorem characterizing its possible complexity. In Section 6 the construction of Section 4 is generalized and the possible complexity of the uniform homeomorphism relations for the resulting classes is completely determined. Finally in Section 7 we prove Theorem 1.3.

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2. Preliminaries on the Borel reducibility hierarchy

In this section we review the Borel reducibility hierarchy of analytic equivalence relations for the convenience of the reader. We give the definitions of all equivalence relations mentioned in Figure 1 and recall their characteristic properties. The reader can find more details in the references provided below, or see [Ga2].

The descriptive set theory studies definable sets and relations on Polish spaces. Recall that a *Polish* space is a separable and completely metrizable topological space. Examples of Polish spaces include $\omega = \mathbb{N}$ with the discrete topology, $\mathbb{R}$ with the usual topology, $\mathbb{R}^\omega$ with the product topology, and the Cantor space $2^\omega = \{0,1\}^\omega$. The simplest examples of definable subsets of a Polish space are the Borel sets. Recall that the collection of all *Borel* sets is the smallest σ-algebra containing all open sets. Thus all Borel subsets of a Polish space can also be arranged in a hierarchy according to their *descriptive complexity.* In this hierarchy the simplest ones are open sets and closed sets. On the next level we have the F_σ sets and G_δ sets, which are respectively countable unions of closed sets and countable intersections of open sets. To continue, we call a set $\mathbf{\Sigma}^0_3$ if it is a countable union of G_δ sets, and $\mathbf{\Pi}^0_3$ if it is a countable intersection of F_σ sets. In general, one can define the classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ in the same fashion for all countable ordinals α. However, in this paper we will not deal with any set beyond $\mathbf{\Sigma}^0_3$ and $\mathbf{\Pi}^0_3$.

It is well-known that $\mathbf{\Sigma}^0_3$ and $\mathbf{\Pi}^0_3$ are distinct classes. To prove that a given subset A of a Polish space X is not $\mathbf{\Sigma}^0_3$, the usual method is to try and show that A is $\mathbf{\Pi}^0_3$-*hard*, that is, given any $\mathbf{\Pi}^0_3$ subset B of 2^ω, there is a continuous function $f : 2^\omega \to X$ such that $B = f^{-1}(A)$. If A is $\mathbf{\Pi}^0_3$ and $\mathbf{\Pi}^0_3$-hard then it is said to be $\mathbf{\Pi}^0_3$-*complete.* For more on this topic see Section 22, [K].

The Borel structure thus given by a Polish topology is known as a *standard Borel structure.* A Borel space (that is, a space with a distinguished σ-algebra of subsets) is called a *standard Borel space* if its Borel structure is standard, that is, induced by an underlying Polish topology. A function f between Polish spaces (or standard Borel spaces) is *Borel* if $f^{-1}(A)$ is Borel for any Borel set A. Any two uncountable standard Borel space are Borel isomorphic to each other.

Other than the examples of Polish spaces mentioned above, we recall another well-known example of a standard Borel space, the Effros Borel space. Let X be a Polish space and $F(X)$ be the hyperspace of all closed subsets of X. The *Effros Borel structure* is the Borel structure on $F(X)$ generated by basic Borel sets of the form

$$\{F \in F(X) : F \cap U \neq \emptyset\}$$

for some open subset U of X. A Polish topology generating the Effros Borel structure was discovered by Beer [B]. We also recall its definition. Let d be a compatible complete metric on X. For any $x \in X$ and $F \in F(X)$, define

$$d(x, F) = \inf\{d(x, y) \,:\, y \in F\}.$$

Now consider the topology generated by all subbasic open sets of the form

$$\{F \in F(X) \,:\, d(x, F) < a\} \text{ or } \{F \in F(X) \,:\, d(x, F) > a\}$$

for some $x \in X$ and $a \in \mathbb{R}$. This topology is known as the Wijsman topology on $F(X)$; Hess [He] observed that it generates the Effros Borel structure, and Beer [B] proved that it is Polish.

The next level of definable subsets beyond Borel subsets of a Polish space consists of analytic ones. Recall that a subset of a Polish space is *analytic* (or $\mathbf{\Sigma}^1_1$) if it is the continuous image of a Polish space. It is well-known that every Borel set is analytic. For more information on Polish spaces, Borel and analytic subsets, and Borel functions, see [K].

Let X be a Polish space and E an equivalence relation on X. We say that E is *analytic* if E is an analytic subset of $X \times X$. Similarly we also speak of *Borel* equivalence relations, or even F_σ, G_δ, $\mathbf{\Sigma}^0_3$, $\mathbf{\Pi}^0_3$ equivalence relations respectively.

The notion of Borel reducibility defined below is fundamental in the theory of equivalence relations as it explores the relative *structural complexity* of equivalence relations. Let X, Y be Polish spaces and E, F be equivalence relations on X, Y, respectively. We say that E is *Borel reducible* to F, and denote it by $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that for all $x_1, x_2 \in X$,

$$x_1 E x_2 \iff f(x_1) F f(x_2).$$

If $E \leq_B F$, then intuitively E is *no more complex than* F, since any complete invariants for the F-equivalence classes can be composed with f to obtain complete invariants for the E-equivalence classes. In the case of both $E \leq_B F$ and $F \leq_B E$, then we denote $E \sim_B F$ and say that E and F are *Borel bireducible.* If $E \sim_B F$, then intuitively E and F have *the same complexity.*

Next we define the benchmark equivalence relations in Figure 1.

(1) The equivalence relation $\mathrm{id}(2^\omega)$ is the identity (or equality) relation on the Cantor space 2^ω, that is, $(x, y) \in \mathrm{id}(2^\omega)$ iff $x = y$. Since all uncountable standard Borel spaces are Borel isomorphic to each other, this relation is Borel bireducible with the identity relation on any uncountable Polish (or standard Borel) space. An equivalence relation that is Borel reducible to $\mathrm{id}(2^\omega)$ is said to be *smooth* or *concretely classifiable*, since it is possible to assign a concrete real number as a complete invariant for each of its equivalence classes.

(2) The equivalence relation E_0 is the eventual agreement relation on 2^ω. In symbols, for $x = (x_n), y = (y_n) \in 2^\omega$,

$$x E_0 y \iff \exists m \in \omega\ \forall n \geq m\ x_n = y_n.$$

In the Borel reducibility hierarchy for Borel equivalence relations E_0 is the minimum one beyond $\mathrm{id}(2^\omega)$ [HaKL].

(3) The equivalence relation E_1 is the eventual agreement relation for countable sequences of real numbers. In symbols, for $\vec{x} = (x_n), \vec{y} = (y_n) \in \mathbb{R}^\omega$,

$$\vec{x} E_1 \vec{y} \iff \exists m \in \omega\ \forall n \geq m\ x_n = y_n.$$

It is easy to see that $E_0 \leq_B E_1$. In the definition of E_1 the space $\mathbb{R}$ can be replaced by the Cantor space 2^ω or the Baire space ω^ω without affecting the complexity of the resulting equivalence relation, since $\mathbb{R}$ is Borel isomorphic to any uncountable Polish space. We will use the alternate versions of the definition in this paper without further elaborations. In the Borel reducibility hierarchy E_1 is an immediate successor of E_0 ([KL]), that is, if $E \leq_B E_1$ then E is Borel bireducible with either E_1 or E_0, or else E is smooth.

(4) For $1 \leq p \leq \infty$ the equivalence relation E_{ℓ_p} is defined on $\mathbb{R}^\omega$ as follows: for $\vec{x} = (x_n), \vec{y} = (y_n) \in \mathbb{R}^\omega$,

$$\vec{x} E_{\ell_p} \vec{y} \iff \vec{x} - \vec{y} \in \ell_p.$$

When there is no danger of confusion we simply use ℓ_p to denote the equivalence relation E_{ℓ_p}. Dougherty and Hjorth [DH] showed that for $1 \leq p \leq q < \infty$, $\ell_p \leq_B \ell_q$. The first author [Ga1] extended this to include the case $q = \infty$. It is also known that $E_1 \leq_B \ell_\infty$ [Ga1] and $E_1 \not\leq_B \ell_p$ for $p < \infty$ [KL].

The equivalence relation ℓ_∞ is perhaps the most important equivalence relation for this paper. Rosendal [Ro] showed that it is a complete K_σ equivalence relation, that is, for any equivalence relation E on a Polish space X, if E is K_σ (that is, a countable union of compact subsets of X^2), then $E \leq_B \ell_\infty$. In particular, if every E-equivalence class is countable, then $E \leq_B \ell_\infty$.

(5) An equivalence relation is called *countable* if every of its equivalence classes is countable. Among all countable Borel equivalence relations there exists a maximum one in terms of Borel reducibility [DJK]. We denote any such equivalence relation by E_∞. By the remark above, $E_\infty \leq \ell_\infty$. An equivalence relation E is *essentially countable* if $E \leq_B E_\infty$.

(6) The equivalence relation $=^+$ codes the equality relation for countable sets of real numbers. In symbols, for $\vec{x} = (x_n), \vec{y} = (y_n) \in \mathbb{R}^\omega$,

$$\vec{x} =^+ \vec{y} \iff \{x_n : n \in \omega\} = \{y_n : n \in \omega\}.$$

It is an easy consequence of a classical theorem of Luzin and Novikov (see Theorem 18.10, [K]) in descriptive set theory that $E_\infty \leq_B =^+$. It is also known that $\ell_\infty \not\leq_B =^+$ (by results of Kechris and Louveau [KL]) and $=^+ \not\leq_B \ell_\infty$ (see below).

(7) The equivalence relation E_0^ω is defined on $(2^\omega)^\omega$ as follows: for $\vec{x} = (x_n), \vec{y} = (y_n) \in (2^\omega)^\omega$,

$$\vec{x} E_0^\omega \vec{y} \iff \forall n \in \omega\ x_n E_0 y_n.$$

E_0^ω has been studied explicitly or implicitly in, for example, [So], [HK], and [HjKL]. Note that it is a $\mathbf{\Pi}_3^0$ equivalence relation. Results of Solecki [So] imply that E_0^ω is not essentially countable, that is, $E_0^\omega \not\leq_B E_\infty$. Further results of Hjorth, Kechris, and Louveau [HK] [HjKL] imply that E_0^ω is not Borel reducible to any $\mathbf{\Sigma}_3^0$ equivalence relations. Thus in particular $E_0^\omega \not\leq_B \ell_\infty$. It is a somewhat elusive task to trace the references for this result; for the convenience of the reader we will give a direct proof of it later in this section.

The importance of E_0^ω lies in both the fact that it is combinatorially easy to analyze and the speculation that it seems to be the simplest (or even minimum) equivalence relation not reducible to ℓ_∞. For instance, it is relatively simple to show that $E_0^\omega \leq_B =^+$ (we will give a proof later in this section); therefore it follows immediately that $=^+ \not\leq_B \ell_\infty$. Also, when we need to consider equivalence relations which seem to be more complex than ℓ_∞, the reducibility of E_0^ω to them gives good test questions.

(8) The equivalence relation E_G^∞ is the universal orbit equivalence relation induced by Borel actions of Polish groups. We shall not deal with general orbit equivalence relations in this paper. Therefore we will omit the details of the definition of E_G^∞. The interested reader can find more information in [BK], [GK], or [Me].

(9) Among all analytic equivalence relations on Polish spaces there is a universal one, that is, every other analytic equivalence relation is Borel reducible to it. Following [LR] we denote it by $E_{\mathbf{\Sigma}_1^1}$. As we mentioned in the Introduction this equivalence relation plays an important role when natural equivalence relations between separable Banach spaces are considered. However, results in this paper only involve equivalence relations much less complex than $E_{\mathbf{\Sigma}_1^1}$.

In the rest of the paper we will consider more equivalence relations, but most of them will be closely related to ℓ_∞.

In the remainder of this section we give some proofs of facts related to E_0^ω (see (7) above) for the convenience of the reader. We fix some notation to be used in these proofs, as well as in the rest of the paper. First, we fix once and for all a computable bijection $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$. Next, we let $\omega^{<\omega}$ denote the countable set of all finite sequences of natural numbers. For $s \in \omega^{<\omega}$ let $|s|$ denote the length of s, that is, if $s = (s_1, \dots, s_n)$ then $|s| = n$. The empty sequence is denoted $\emptyset$, and we set $|\emptyset| = 0$. If $s = (s_1, \dots, s_n), t = (t_1, \dots, t_m) \in \omega^{<\omega}$, then we let

$$s * t = \begin{cases} (s_1, \dots, s_n, t_{n+1}, \dots, t_m), & \text{if } m > n, \\ s, & \text{if } m \leq n. \end{cases}$$

Then $|s * t| = \max\{|s|, |t|\}$, and $s * t$ is obtained by replacing the first $|s|$ many elements of t by s. This definition also makes sense when $t \in \omega^{<\omega}$ is replaced by an element of ω^ω. For s, t as above we also let

$$s \oplus t = (s_1, t_1, s_2, t_2, \dots).$$

Then $|s \oplus t| = |s| + |t|$. This definition makes sense when both s and t are replaced by elements of ω^ω. For $x, y \in \omega^\omega$, $x \oplus y$ is obtained from shuffling the elements of x and y into a single sequence.

Lemma 2.1. $E_0^\omega \leq_B =^+$.

Proof. Let $s^0, s^1, \dots$ be an enumeration of $\omega^{<\omega}$. Fix some $\vec{z} = (z_n) \in (2^\omega)^\omega$ such that for all $i \neq j \in \omega$, $(z_i, z_j) \notin E_0$. For $\vec{x} = (x_n) \in (2^\omega)^\omega$, let $f(x) = (y_n)$, where for $n = \langle i, j \rangle$,

$$y_n = z_i \oplus (s^j * x_i).$$

It is easy to verify that f is a Borel (in fact continuous) reduction from E_0^ω to $=^+$. $\square$

Lemma 2.2. *E_0^ω is not Borel reducible to any $\mathbf{\Sigma}_3^0$ equivalence relation.*

Proof. Suppose X is a Polish space, E a $\mathbf{\Sigma}_3^0$ equivalence relation on X, and $f\colon (2^\omega)^\omega \to X$ a Borel function such that for all $\vec{x}, \vec{y} \in (2^\omega)^\omega$,

$$\vec{x} \, E_0^\omega \, \vec{y} \iff f(x) \, E \, f(y).$$

Since f is Borel, and hence Baire measurable, there is a comeager set $C \subseteq (2^\omega)^\omega$ such that $f \restriction C$ is continuous. We may assume C is a G_δ set. We may now compute $E_0^\omega \cap (C \times C)$ to be $\mathbf{\Sigma}_3^0$, namely,

$$(x, y) \in E_0^\omega \cap (C \times C) \iff x, y \in C \ \wedge \ (f(x), f(y)) \in E.$$

Since $f \restriction C$ is continuous and $E \in \mathbf{\Sigma}_3^0$, this shows $E_0^\omega \cap (C \times C)$ to be $\mathbf{\Sigma}_3^0$. To get a contradiction it thus suffices to prove the following claim.

Claim. For every comeager set $C \subseteq (2^\omega)^\omega$, $E_0^\omega \cap (C \times C)$ is $\mathbf{\Pi}_3^0$-complete.

Proof. Let $B = \{x \in 2^\omega : \forall i \in \omega \ \exists j \in \omega \ \forall k \geq j \ x(\langle i, k \rangle) = 1\}$. Then B is clearly $\mathbf{\Pi}_3^0$. We first show that B is $\mathbf{\Pi}_3^0$-complete. For this, let $A \subseteq 2^\omega$ be $\mathbf{\Pi}_3^0$, say $A = \bigcap_i \bigcup_j \bigcap_l A_{i,j,k}$, where each $A_{i,j,k}$ is clopen. Define $\rho\colon 2^\omega \to 2^\omega$ as follows. For each $i, k \in \omega$ let $a_{x,i,k} \in \omega$ be the least integer $j \leq k$, if one exists, such that $x \in A_{i,j,k'}$ for all $k' \leq k$. Let $\rho(x)(i, k) = 1$ iff $a_{x,i,k}$ and $a_{x,i,k-1}$ are both defined and are equal. Otherwise set $\rho(x)(i, k) = 0$. The map ρ is continuous from 2^ω to 2^ω, and $x \in A$ iff $\rho(x) \in B$. Thus, B is $\mathbf{\Pi}_3^0$-complete.

Note that $(2^\omega)^\omega$ is homeomorphic to $2^{\omega \times \omega}$. For notational simplicity we work with $2^{\omega \times \omega}$ below, and identify it with $(2^\omega)^\omega$. If $s \in 2^{n \times m}$ for some $n, m \in \omega$, then the basic clopen set determined by s, denoted by N_s, is the set $\{x \in 2^{\omega \times \omega} : \forall i < n, j < m \ x(i, j) = s(i, j)\}$. Write $C = \bigcap_n D_n$ where each D_n is open dense in $2^{\omega \times \omega}$.

We next define two continuous functions $\pi_1, \pi_2 \colon 2^\omega \to 2^{\omega\times\omega}$ so that

$$x \in B \iff (\pi_1(x), \pi_2(x)) \in E_0^\omega \restriction (C \times C).$$

For each sequence $s \in 2^n$ we will define values $\pi_1(s), \pi_2(s) \in 2^{p(n)\times p(n)}$ for some $p = p(n)$ which depends only on n. We will then take, for $x \in 2^\omega$, $\pi_1(x) = \bigcup_n \pi_1(x \restriction n)$ and likewise for $\pi_2(x)$.

Suppose inductively that for some $n \in \omega$ and every sequence $s \in 2^n$ we have defined $\pi_1(s), \pi_2(s) \in 2^{p\times p}$ for some $p = p(n) \in \omega$ which depends only on n. Suppose also that $N_{\pi_1(s)}, N_{\pi_2(s)} \subseteq D_n$ for each $s \in 2^n$. Let $n+1 = \langle i, k\rangle$. For each $s' \in 2^{n+1}$ extending $s \in 2^n$, extend $t_1 := \pi_1(s)$ and $t_2 := \pi_2(s)$ to t_1', t_2' by letting $t_1'(i, p(n)+k) = t_2'(i, p(n)+k) = 1$ if $s(n) = 1$, and otherwise setting $t_1'(i, p(n)+k) = 0$, $t_2'(i, p(n)+k) = 1$. Extend t_1', t_2' to t_1'', t_2'' in $2^{q_n\times q_n}$, where $q_n = p(n)+n$, by setting all other undefined values to 0. Note that all of the t_1'', t_2'' are elements of $2^{q_n\times q_n}$. Let $p(n+1)$ be large enough so that there is a finite function $h_{n+1} \colon (p(n+1)\times p(n+1)) - (q_n \times q_n) \to \{0,1\}$ such that for all of the t_1'', t_2'' we have that $u_1 = t_1'' \cup h_{n+1}$ and $u_2 = t_2'' \cup h_{n+1}$ determine basic open sets with $N_u \subseteq D_{n+1}$. We can achieve this in $2 \cdot 2^{n+1}$ steps, using the fact that D_{n+1} is dense open. Set $\pi_1(s') = u_1$, $\pi_2(s') = u_2$. Note that for any $s_1, s_2 \in 2^{n+1}$ and $a, b \in \{1,2\}$, $\pi_a(s_1)$, $\pi_b(s_2)$ differ in at most one point of $(p(n+1)\times p(n+1)) - (p(n)\times p(n))$.

Clearly π_1, π_2 are continuous and $\pi_1(x)$, $\pi_2(x) \in C$ for any $x \in 2^\omega$. If $x \in B$, then for each i let $k(i)$ be such that $x(\langle i,k\rangle) = 1$ for all $k \geq k(i)$. Fix $i \in \omega$. For any $n \geq \langle i, k(i)\rangle$, if $n = \langle i,j\rangle$ for some j then $\pi_1(x \restriction n)$, $\pi_2(x \restriction n)$ are extended identically in going to $\pi_1(x \restriction n+1)$ and $\pi_2(x \restriction n+1)$ (namely, they have value 1 at $(i, p(n)+j)$ and 0 at the other new points of the domain). If $n = \langle i', j\rangle$ where $i' \neq i$, then we still have that $\pi_1(x \restriction n)$, $\pi_2(x \restriction n)$ are extended identically on point of the form (i,k) (they both are 0 there). So, $\pi_1(x)$, $\pi_2(x)$ agree at coordinates of the form (i,k) for all large enough k. So, $(\pi_1(x)\, \pi_2(x)) \notin E_0^\omega$.

Conversely, if $x \notin B$, then for some i, there are infinitely many j that $x(\langle i,j\rangle) = 0$. Fix such an i. For each j with $x(\langle i,j\rangle) = 0$ let $n = \langle i,k\rangle$, and we have that $\pi_1(x)(n)$ and $\pi_2(x)(n)$ disagree at $(i, p(n)+j)$. This implies that $\neg \pi_1(x)\, E_0^\omega\, \pi_2(x)$. □

This completes the proof of lemma 2.2. □

3. CODINGS OF SEPARABLE BANACH SPACES AND THE LOCAL EQUIVALENCE

To apply the descriptive set theoretic framework to the study of equivalence relations on separable Banach spaces, the collection of separable Banach spaces must be viewed as a Polish space.

One way to do this is to use the well-known theorem of Banach and Mazur that $C[0,1]$ is a universal separable Banach space, that is, every separable Banach space is linearly isometric to a (necessarily closed) subspace of $C[0,1]$. The collection of all separable Banach spaces is then viewed as a subspace of the hyperspace $F(C[0,1])$ with the Wijsman topology (see Section 2). Let

$$\mathfrak{B} = \{F \in F(C[0,1]) \,:\, F \text{ is a linear subspace of } C[0,1]\,\}.$$

We check below that $\mathfrak{B}$ is a Polish subspace.

Lemma 3.1. *$\mathfrak{B}$ is a G_δ subspace of $F(C[0,1])$, hence is Polish.*

Proof. Fix a countable dense $D \subseteq C[0,1]$. Let d be the metric on $C[0,1]$ given by its norm. We claim that for any $F \in F(C[0,1])$, $F \in \mathfrak{B}$ iff

$$\forall p,q,a,b \in \mathbb{Q}\, \forall x,y \in D\, [\, d(x,F) < a \wedge d(y,F) < b \Longrightarrow d(px+qy,F) < |pa|+|qb|\,].$$

If F is a linear subspace of $C[0,1]$ the demonstrated condition clearly holds. Conversely, suppose the condition holds. Since F is closed, it suffices to show that for all $u,v \in F$ and $p,q \in \mathbb{Q}$, $pu+qv \in F$. For this take two sequences $x_n, y_n \in D$ such that $d(x_n,u), d(y_n,v) < 2^{-n}$. Then $d(px_n+qy_n,F) < (|p|+|q|)2^{-n}$ by the assumption, and $d(px_n+qy_n, pu+qv) < (|p|+|q|)2^{-n}$. Thus $d(pu+qv,F) < (|p|+|q|)2^{-n+1}$. Since n is arbitrary, we have that $d(pu+qv,F) = 0$ and $pu+qv \in F$. The claim implies immediately that $\mathfrak{B}$ is G_δ in the Wijsman topology of $F(C[0,1])$. □

In discussing Banach spaces with a distinguished Schauder basis another representation is often used. A fundamental result of Pełczyński [P] says that there is a universal basis $U = (e_i)$, that is, for every separable Banach space with a basis (X,B), where $B = (x_i)$, there is a one-to-one map $f\colon B \to U$ which extends to a linear isomorphism from X to the space spanned by $f(B)$. In this manner, the collection of separable Banach spaces with a basis can be identified with the Cantor space 2^ω. For $x \in 2^\omega$, let X_x be the separable Banach space with a basis coded by x. The space of all Banach spaces with infinite bases correspond to $[\omega]^\omega$, the set of all infinite subsets of 2^ω, which is a G_δ subset of 2^ω and a Polish space in its own right.

In practice it is often easier to work with the following direct coding for Banach spaces with infinite bases. Fix once and for all for the rest of the paper an enumeration (s^n) of $\mathbb{Q}^{<\omega}$, the set of all finite sequences of rational numbers. To any separable Banach space with a infinite basis $(Y,(y_i))$, we associate a sequence of real numbers $(r_n) \in \mathbb{R}^\omega$, where

$$r_n = \Big\|\sum_{i=1}^{k} a_i^n y_i\Big\|_Y$$

if $s^n = (a_1^n, \ldots, a_k^n)$. Recall that a normalized (i.e., all y_i have norm 1) basis (y_i) is called *monotone* if the projections onto initial segments of (y_i) have norm 1. There is no loss of generality in restricting to monotone bases, since for every normalized basis we can take an equivalent norm for which the basis is monotone. Let $\mathfrak{B}_b \subseteq \mathbb{R}^\omega$ be the set of all possible sequences (r_n) associated with Banach spaces with a monotone basis.

Again we check below that $\mathfrak{B}_b$ is a Polish space. We henceforth use the phrase "Banach space with basis" to denote a pair (X, B), where X is a Banach space, and B is a basis. It is these objects that are coded by the reals in $\mathfrak{B}_b$.

Lemma 3.2. *$\mathfrak{B}_b$ is a closed subspace of $\mathbb{R}^\omega$, hence is Polish.*

Proof. For notational convenience in this proof we identify $s^n = (a_1^n, \dots, a_k^n)$ with the infinite sequence $(a_1^n, \dots, a_k^n, 0, 0, \dots)$. Then it makes sense to speak of $s^n + s^m \in \mathbb{Q}^{<\omega}$ for any $n, m \in \omega$, and $ps^n \in \mathbb{Q}$ for any $p \in \mathbb{Q}$ and $n \in \omega$. Now for any $(r_n) \in \mathbb{R}^\omega$, $(r_n) \in \mathfrak{B}_b$ iff all of the following hold:

(i) if $s^n = (0, \dots, 0, 1, 0. \dots, 0)$, then $r_n = 1$;
(ii) if s^m coincides with an initial segment of s^n, then $r_m \le r_n$.
(iii) if $s^n + s^m = s^l$, then $r_l \le r_n + r_m$;
(iv) for any $p \in \mathbb{Q}$, if $s^m = ps^n$, then $r_m = |p| r_n$.

The conditions listed are closed for (r_n) in $\mathbb{R}^\omega$. Note that (i) and (ii) imply that the basis is monotone which implies that any non-zero linear combination of the y_i has positive norm. □

Given any $(r_n) \in \mathfrak{B}_b$, by the proof of Lemma 3.2 we can associate a Banach space with an infinite basis whose norm function is approximated by the sequence (r_n). In this manner each element of $\mathfrak{B}_b$ codes a Banach space with basis. For $y \in \mathfrak{B}_b$, let Y_y be the space coded by y.

We remark that the two codings for Banach spaces with bases are equivalent in the following precise sense. It is easy to see that there is a continuous function $\varphi\colon [\omega]^\omega \to \mathfrak{B}_b$ such that for all $x \in [\omega]^\omega$, X_x is linearly isometric to $Y_{\varphi(x)}$. Conversely, by the proof of Pełczyński's theorem [P] there is also a Borel function $\psi\colon \mathfrak{B}_b \to [\omega]^\omega$ such that for all $y \in \mathfrak{B}_b$, Y_y is linearly isomorphic to $X_{\psi(y)}$.

As for the relationship between codings using elements of $\mathfrak{B}$ versus those of $\mathfrak{B}_b$, we denote by $\mathfrak{B}_\beta$ the subspace of $\mathfrak{B}$ consisting of all linear subspaces of $C[0,1]$ admitting bases. It follows immediately from the proof of the Banach-Mazur theorem that there is a Borel function $\Phi : \mathfrak{B}_b \to \mathfrak{B}_\beta \subseteq \mathfrak{B}$ such that for all $y \in \mathfrak{B}_b$, Y_y is linearly isometric to $\Phi(y)$. Intuitively, in defining $\Phi(y)$ one omits the given basis and obtains an isomorphic (in fact isometric) copy of the space as a subspace of $C[0,1]$. It is easy to see that $\mathfrak{B}_\beta$ coincides with the isomorphic saturation of $\Phi(\mathfrak{B}_b)$, denoted $[\Phi(\mathfrak{B}_b)]$, which is also the same as the isometric saturation of $\Phi(\mathfrak{B}_b)$. Obviously both $\Phi(\mathfrak{B}_b)$ and $\mathfrak{B}_\beta = [\Phi(\mathfrak{B}_b)]$ are analytic subsets of $\mathfrak{B}$. However, it is not known whether either of them is Borel.

Rosendal has pointed out that the function Φ can be improved to be injective, that is, there is a Borel *injective* function Ψ with all the above properties. To see this, fix $\lambda : \mathfrak{B}_b \to [0,1]$ a Borel injection and $\varphi : C[0,1] \oplus_\infty C[0,1] \to C[0,1]$ a linear isometric embedding. For any $y \in \mathfrak{B}_b$, let

$$\Psi(y) = \{\, \varphi(v, \lambda(y)v) \in C[0,1] \,:\, v \in \Phi(y) \subseteq C[0,1] \,\}.$$

Then $\Psi(y)$ and $\Phi(y)$ are linearly isometric, and Ψ is obviously injective because of the injectivity of λ and φ. It follows that $\Psi(\mathfrak{B}_b)$ is Borel. Note that $\mathfrak{B}_\beta = [\Psi(\mathfrak{B}_b)] = [\Phi(\mathfrak{B}_b)]$ and the question about its Borelness remains unresolved.

Next we turn to equivalence relations between separable Banach spaces.

We remark first that the uniform homeomorphism relation is analytic as an equivalence relation on either $\mathfrak{B}$ or $\mathfrak{B}_b$. This was noted in [FLR], and in fact it it proved there that the uniform homeomorphism relation on all Polish metric spaces is complete analytic. For the convenience of the reader we recall the following argument. Let $\approx$ denote the uniform homeomorphism relation on $\mathfrak{B}$. Then for $X, Y \in \mathfrak{B}$, $X \approx Y$ iff there exist $(x_n), (y_n) \in C[0,1]^\omega$ such that

(a) $x_n \in X$ and $y_n \in Y$ for all $n \in \omega$;
(b) the sets $D_X := \{x_n : n \in \omega\}$ and $D_Y := \{y_n : n \in \omega\}$ are dense in X and Y respectively;
(c) the map $f : D_X \to D_Y$ with $f(x_n) = y_n$ for all $n \in \omega$ is a uniformly continuous bijection, with f^{-1} also uniformly continuous.

One direction of the equivalence is clear. For the other direction, we note that the uniform homeomorphism f defined on a dense set D_X can be uniquely extended to a necessarily uniform homeomorphism of the entire space, since Cauchy sequences in D_X will correspond to Cauchy sequences in D_Y by the uniform continuity of f and f^{-1}. Now the conditions (a) through (c) are all Borel conditions for $X, Y, (x_n)$, and (y_n). Hence $\approx$ is analytic. It also follows immediately that the uniform homeomorphism relation on $\mathfrak{B}_b$ is analytic via the pullback of the Borel function Φ defined above.

In the remainder of this section we define a notion of local equivalence inspired by Ribe's theorem [Ri] and study its basic properties. In doing this we recall some concepts and results from Banach space theory. All other unexplained terms and facts can be found in [BL] or [T].

Recall that, for linearly isomorphic Banach spaces X and Y, the *Banach-Mazur distance* between X and Y is defined as

$$d(X,Y) := \inf\{\, \|T\|\|T^{-1}\| \,:\, T : X \to Y \text{ is an isomorphism}\,\}.$$

The following theorem is a fundamental result about uniform homeomorphism.

Theorem 3.3 (Ribe [Ri])**.** *If X and Y are uniformly homeomorphic Banach spaces, then there exists a constant $C > 0$ such that for every finite-dimensional subspace E of X there exists a finite-dimensional subspace F of Y such that $d(E,F) \leq C$, and vice versa.*

This motivates the following concept.

Definition 3.4. Let X and Y be Banach spaces. We say that X and Y are *locally equivalent*, and denote by $X \equiv_L Y$, if there exists a constant

$C > 0$ such that for every finite-dimensional subspace E of X there exists a finite-dimensional subspace F of Y such that $d(E, F) \leq C$, and vice versa.

Here we refer to the structure of finite-dimensional subspaces of a Banach space as its local structure. In the literature the local equivalence between X and Y is sometimes informally referred to as X and Y having the same finite-dimensional subspaces. Ribe's theorem states that uniformly homeomorphic spaces are locally equivalent. The converse is not true. For instance, as we mentioned in the Introduction, ℓ_p is not uniformly homeomorphic to L_p for $1 \leq p < \infty$, $p \neq 2$; however, they are locally equivalent.

In the following we compute the descriptive complexity of the local equivalence as an equivalence relation on either the Polish space $\mathfrak{B}$ of all separable Banach spaces or the Polish space $\mathfrak{B}_b$ of all separable Banach spaces with basis.

Lemma 3.5. *Local equivalence is a $\mathbf{\Sigma}^0_3$ equivalence relation on either $\mathfrak{B}$ or $\mathfrak{B}_b$.*

Proof. First consider $\equiv_L$ as an equivalence relation on $\mathfrak{B}_b$. Let $(X, (e_i))$ and $(Y, (f_i))$ be the Banach spaces with basis coded as elements of $\mathfrak{B}_b$ by $x, y \in \mathbb{R}^\omega$. Note that every finite-dimensional subspace of $(X, (e_i))$ can be approximated by a space with a (finite) basis consisting of finite rational linear combinations of the e_i. We use the enumeration $\{s^n\}$ of $\mathbb{Q}^{<\omega}$ in the definition of $\mathfrak{B}_b$. For $s^n = (a^n_1, \ldots, a^n_k) \in \mathbb{Q}^{<\omega}$, let $s^n(X) = \sum_{i=1}^k a^n_i e_i$. For $\vec{n} = n_1, \ldots, n_N \in \omega$, let $X_{\vec{n}}$ be the $\leq N$-dimensional subspace of X with basis $s^{n_1}(X), \ldots, s^{n_N}(X)$. Similarly we define $s^m(Y)$ and $Y_{\vec{m}}$ for $\vec{m} = m_1, \ldots, m_M \in \omega$.

Let I be the set of $\vec{n} = (n_1, \ldots, n_N) \in \mathbb{N}^{<\omega}$ such that the vectors $s^{n_1}, \ldots, s^{n_N}$ are linearly independent. Note that if $X \in \mathfrak{B}_b$ is coded by x, and $\vec{n} \in I$, then the vectors $s^{n_1}(X), \ldots, s^{n_N}(X)$ are linearly independent in the space X (since x codes a monotone basis for X).

With this notation we have that $X \equiv_L Y$ iff

$$\exists M \geq 1 \ \forall N \geq 1 \, \forall (n_1, \ldots, n_N) \in I \, \exists (m_1, \ldots, m_N) \in I \, d(X_{n_1,\ldots,n_N}, Y_{m_1,\ldots,m_N}) < M$$

and vice versa. It suffices to show that for fixed $\vec{n}$ and $\vec{m}$ that the relation on $\mathfrak{B}_b$ given by

$$U(x, y) \Leftrightarrow d(X^x_{n_1,\ldots,n_N}, Y^y_{m_1,\ldots,m_N}) < M$$

is open, where X^x and Y^y denote the spaces coded by x and y. Fix x, y with $U(x, y)$. Let $T \colon X^x_{\vec{n}} \to Y^y_{\vec{m}}$ be a linear isomorphism with $\|T\|\|T^{-1}\| < M$. Let $x_1, \ldots, x_N$ denote $s^{n_1}(X), \ldots, s^{n_N}(X)$, and let $y_1 = T(x_1), \ldots, y_N = T(x_N)$. For $x' \in \mathfrak{B}_b$, let $x'_1, \ldots, x'_N$ denote $s^{n_1}(X'), \ldots, s^{n_N}(X')$ where X' is the space coded by x'. Let $T' \colon X'_{\vec{n}} \to Y$ be the linear map defined by $T'(x'_1) = y_1, \ldots, T'(x'_N) = y_N$.

It suffices by symmetry to show that for any $\epsilon > 0$ there is an open $V \subseteq \mathbb{R}^\omega$ containing x such that for all $x' \in \mathfrak{B}_b \cap V$ we have $|\|T\| - \|T'\|| < \epsilon$. Let $\rho > 0$ be such that $\rho < \min\{\|x_i\|\} \leq \max\{\|x_i\|\} < \frac{1}{\rho}$. Let $0 < \eta < \inf\{\|\alpha_1 x_1 +$

$\cdots + \alpha_N x_N\|_X \colon \vec{\alpha} \in S_N\}$, where $S_N = \{\vec{\alpha}\colon \sum \alpha_i^2 = 1\}$. By definition of the product topology on $\mathbb{R}^\omega$, there is clearly an open set V_1 about x such that for $x' \in V_1$ we have $\rho < \min\{\|x_i'\|\} \leq \max\{\|x_i'\|\} < \frac{1}{\rho}$. It thus suffices to show that for all $\epsilon > 0$ there is a neighborhood $V \subseteq V_1$ of x such that for all $x' \in V$ we have that $|\|\alpha_1 x_1 + \cdots + \alpha_N x_N\|_X - \|\alpha_1 x_1' + \cdots + \alpha_N x_N'\|_{X'}| < \epsilon\left(\frac{\eta^2\rho}{2N\|T\|}\right)$, for all $\vec{\alpha} \in S_N$. For then, letting $v = \alpha_1 x_1 + \cdots + \alpha_N x_N$ and $v' = \alpha_1 x_1' + \cdots + \alpha_N x_N'$ we have (noting $T(v) = T(v')$ and assuming $\epsilon < \frac{\eta}{2}$):

$$|\frac{\|T(v)\|}{\|v\|} - \frac{\|T(v')\|}{\|v'\|}| = \frac{\|T(v)\|}{\|v\|\|v'\|}(|\|v\| - \|v'\||) \leq \frac{\|T\|N}{\rho}\frac{2}{\eta^3}(|\|v\| - \|v'\||) \leq \epsilon$$

Let $\mathfrak{N} \subseteq S_N \cap \mathbb{Q}^N$ be such that for all $\vec{\alpha} \in S_N$, there is a $\vec{q} \in \mathfrak{N}$ such that $|\alpha_i - q_i| < \frac{\epsilon\rho\delta}{3N}$ for all i, where $\delta = \frac{\eta^2\rho}{2N\|T\|}$. (the rational points on S_N are dense). Given $\vec{\alpha} \in S_N$, let $\vec{q} \in \mathfrak{N}$ be such that $|\alpha_i - q_i| < \frac{\epsilon\rho\delta}{3N}$ for all $1 \leq i \leq N$. We have that $|\|\sum \alpha_i x_i\| - \|\sum q_i x_i\|| < (\frac{\epsilon\rho\delta}{3N})(N)\max\{\|x_i\|\} \leq \frac{\epsilon\delta}{3}$, with a similar estimate for the x_i'. Since the q_i and s^{n_i} are rational, if V is a small enough neighborhood of x and $x' \in V$ we will have $|\|\sum q_i x_i\| - \|\sum q_i x_i'\|| < \frac{\epsilon\delta}{3}$. Thus, $|\|\sum \alpha_i x_i\| - \|\sum \alpha_i x_i'\|| < \epsilon\delta$.

Next consider $\equiv_L$ as an equivalence relation on $\mathfrak{B}$. Fix a countable dense $D \subseteq C[0,1]$. For this part of the proof let d be the metric on $C[0,1]$ given by the norm. Let Q be the set of all quadruples (s,t,n,q) such that

1. $s, t \in D^{<\omega}$ (the set of all finite sequences of elements of D), $n \in \omega$, $q \in \mathbb{Q}$;
2. there is some $k \in \omega$ such that $s, t \in D^k$, that is, $|s| = |t| = k$;
3. if $s = (s_1, \ldots, s_k)$, $t = (t_1, \ldots, t_k)$, then for any $x_1, \ldots, x_k, y_1, \ldots, y_k \in C[0,1]$ such that $d(s_i, x_i), d(t_i, y_i) < 2^{-n}$ for $1 \leq i \leq k$, letting T be the linear map from $\mathrm{span}(x_1, \ldots, x_k)$ to $\mathrm{span}(y_1, \ldots, y_k)$ with $T(x_i) = y_i$ for $1 \leq i \leq k$, we have that $\|T\|\|T^{-1}\| < q$.

Note that if $x_1, \ldots, x_k, y_1, \ldots, y_k \in C[0,1]$ and the linear map $T : \mathrm{span}(x_1, \ldots, x_k) \to \mathrm{span}(y_1, \ldots, y_k)$ sending x_i to y_i satisfies $\|T\|\|T^{-1}\| < C$ for some $C > 0$, then there is a quadruple $(s,t,n,q) \in Q$ such that $q < C$ and for all $1 \leq i \leq k$, $d(s_i, x_i), d(t_i, y_i) < 2^{-n}$.

We claim that for any $X, Y \in F(C[0,1])$, $X \equiv_L Y$ iff

$$\begin{aligned}
&\exists C \in \mathbb{Q}\ \forall k \in \omega\ \forall z_1, \ldots, z_k \in D\ \forall \epsilon \in \mathbb{Q}\\
&\quad \{\ \forall 1 \leq i \leq k\ d(z_i, X) < \epsilon \Longrightarrow \exists (s,t,n,q) \in Q\ [\,q < C\ \wedge\ 2^{-n} < \epsilon \wedge\\
&\qquad \forall 1 \leq i \leq k\ (\,d(s_i, z_i) < \epsilon\ \wedge\ d(t_i, Y) < 2^{-n}\ \wedge\ d(s_i, X) < 2^{-n}\,)\,]\ \}.
\end{aligned}$$

To prove the claim, first suppose $X \equiv_L Y$, and let $C > 0$ be a witness. Suppose $z_1, \ldots, z_k \in D$ and ϵ are given, and for $1 \leq i \leq k$ let $x_i \in X$ be such that $d(x_i, z_i) < \epsilon$. By the local equivalence between X and Y there are $y_1, \ldots, y_k \in Y$ such that the linear map $T : \mathrm{span}(x_1, \ldots, x_k) \to \mathrm{span}(y_1, \ldots, y_k)$ sending x_i to y_i satisfies that $\|T\|\|T^{-1}\| < C$. We have a quadruple $(s,t,n,q) \in Q$ such that $q < C$, $d(s_i, x_i), d(t_i, y_i) < 2^{-n}$ for all

$1\leq i\leq k$. Moreover, we may choose n to be large enough such that $2^{-n}<\epsilon$ and $d(s_i,z_i)<\epsilon$. This verifies the displayed property.

For the other implication, let C be as in the displayed property. Let $x_1,\dots,x_k\in X$ be given. Let $\epsilon>0$ be sufficiently small compared with k. Let $z_1,\dots,z_k\in D$ with $d(z_i,x_i)<\epsilon$ for all $1\leq i\leq k$. Then the displayed property gives a quadruple $(s,t,n,q)\in Q$. Thus $q<C$, $2^{-n}<\epsilon$, and for $1\leq i\leq k$, $d(s_i,z_i)<\epsilon$, $d(t_i,Y)<2^{-n}$, $d(s_i,X)<2^{-n}$. In particular there are $y_1,\dots,y_k\in Y$ such that $d(t_i,y_i)<2^{-n}$ for all $1\leq i\leq k$, and by the definition of Q the linear map $T:\mathrm{span}(s_1,\dots,s_k)\to\mathrm{span}(y_1,\dots,y_k)$ sending s_i to y_i satisfies that $\|T\|\|T^{-1}\|<q$. Since $d(s_i,x_i)<2\epsilon$, and ϵ is sufficiently small, the map $S:\mathrm{span}(x_1,\dots,x_k)\to\mathrm{span}(y_1,\dots,y_k)$ sending x_i to y_i satisfies that $\|S\|\|S^{-1}\|<C+1$.

The displayed property is apparently $\mathbf{\Sigma}^0_3$ in the Wijsman topology on $F(C[0,1])$. It follows that $\equiv_L$ is $\mathbf{\Sigma}^0_3$ on $\mathfrak{B}$. ☐

We can also consider the local equivalence on the space 2^ω of codes for Banach spaces with basis via the Pełczyński universal basis. Recall that there is a continuous function $\varphi:[\omega]^\omega\to\mathfrak{B}_b$ such that for any $x\in 2^\omega$, X_x is linearly isometric to $Y_{\varphi(x)}$. Via this map the local equivalence on $[\omega]^\omega$ is continuously reduced to $\equiv_L$ on $\mathfrak{B}_b$. It follows that the local equivalence on 2^ω is also $\mathbf{\Sigma}^0_3$.

Now it follows from Lemma 2.2 that E_0^ω is not Borel reducible to $\equiv_L$ on either $\mathfrak{B}$ or $\mathfrak{B}_b$, and by Lemma 2.1 $=^+$ is not Borel reducible also to either of them. In Section 5 we will prove that in fact $\equiv_L$ (on either $\mathfrak{B}$ or $\mathfrak{B}_b$) is Borel bireducible to ℓ_∞, thus completely determine its complexity in the Borel reducibility hierarchy.

4. The uniform homeomorphism on a class of Banach spaces

In this section we construct a class of Banach spaces and completely characterize its uniform homeomorphism relation. In the construction and proofs we use a few well-known results in Banach space theory. Our standard reference for undefined terms and unexplained results is [T].

Recall that two given bases (x_i) and (y_i) of Banach spaces are said to be *C-equivalent* for $C>0$ if there exist positive constants A,B with $AB\leq C$ such that for all scalar sequences (a_i),

$$\frac{1}{A}\|\sum_i a_ix_i\|\leq\|\sum_i a_iy_i\|\leq B\|\sum_i a_ix_i\|.$$

We will make use of the following important notion in the study of the local structures of Banach spaces. Let X be a Banach space and let $1\leq p\leq 2$. The *type p constant $T_{p,n}(X)$ of X over n vectors* is the smallest positive number such that for arbitrary n vectors $x_1,\dots,x_n\in X$,

$$\left(\operatorname*{Ave}_{\varepsilon_i=\pm 1}\Big\|\sum_{i=1}^{n}\varepsilon_i x_i\Big\|^2\right)^{1/2} \leq T_{p,n}(X)\big(\sum_{i=1}^{n}\|x_i\|^p\big)^{1/p}. \tag{4.1}$$

X is said to have *type* p if $T_p(X)=\sup_n T_{p,n}(X)<\infty$. For ℓ_q^n spaces type p constants can be easily computed and they satisfy the following estimates

$$n^{\max(0,1/q-1/p)} \quad\leq\quad T_p(\ell_q^n)\leq cq^{1/2}n^{\max(0,1/q-1/p)} \text{ for } 1\leq q<\infty, \tag{4.2}$$

where c is a universal constant. Moreover, the proof of the lower estimate in (4.2) also shows that for $n\leq k$,

$$T_{p,n}(\ell_q^k)\leq cq^{1/2}n^{\max(0,1/q-1/p)}. \tag{4.3}$$

Note that type is a linear notion, in particular, if $T:Y\to X$ is a linear embedding then $T_{p,n}(Y)\leq\|T\|\|T^{-1}\|T_{p,n}(X)$.

For a sequence $\vec{p}=(p_i)\in(1,2)^\omega$ by $S_{\vec{p}}$ we will denote the ℓ_2-direct sum of finite-dimensional $\ell_{p_i}^{n_i}$ spaces for a fixed sequence of increasing dimensions (n_i). That is,

$$S_{\vec{p}}:=\big(\sum_{i=1}^{\infty}\oplus\,\ell_{p_i}^{n_i}\big)_2.$$

The next theorem singles out a class of such spaces on which isomorphism, local equivalence and uniform homeomorphism relations all coincide.

Theorem 4.1. *Let $I_i=[l_i,r_i]$ be a sequence of successive intervals in $(1,2)$. Then there exists $(n_i)\in\omega^\omega$ such that for $\vec{p}=(p_i)$ and $\vec{q}=(q_i)$ with each $p_i,q_i\in I_i$ we have that $S_{\vec{p}}$ is uniformly homeomorphic to $S_{\vec{q}}$ if and only if there exists a constant $C\geq 1$ such that*

$$n_i^{|\frac{1}{p_i}-\frac{1}{q_i}|}\leq C$$

for all $i\in\omega$.

Proof. For any sequence of dimensions (n_i) if $n_i^{|\frac{1}{p_i}-\frac{1}{q_i}|}\leq C$ for some $C<\infty$, then $d(\ell_{p_i}^{n_i},\ell_{q_i}^{n_i})\leq C$ for all $i\in\omega$. In fact, in this case the unit vector bases are C-equivalent. From this it easily follows that $S_{\vec{p}}$ and $S_{\vec{q}}$ are C-isomorphic, and in particular, they are uniformly homeomorphic.

Let $I_i=[l_i,r_i]$ be a sequence of given intervals in $(1,2)$. Pick a sequence (n_i) of natural numbers such that

$$\sup n_i^{\frac{1}{l_i}-\frac{1}{r_i}}=\infty \text{ and } n_{i+1}^{1/r_{i+1}}\geq n_i^{1/l_i},\quad i=1,2,3,\ldots. \tag{4.4}$$

Suppose without loss of generality that $p_i<q_i\in[l_i,r_i]$ with $\sup_i n_i^{1/p_i-1/q_i}=\infty$. By Ribe's theorem 3.3 it is sufficient to show that the spaces in the sequence $(\ell_{p_i}^{n_i})$ do not linearly embed in $S_{\vec{q}}$ with a uniform constant.

Fix $i_0 \in \omega$. Put $n = n_{i_0}$, $p = p_{i_0}$ and let $T : \ell_p^n \to S_{\vec{q}}$ be a linear embedding. Since

$$T_{2,n}(\ell_p^n) \leq \|T\|\|T^{-1}\|T_{2,n}(S_{\vec{q}}),$$

and $T_{2,n}(\ell_p^n) \geq n^{1/p-1/2}$, we need an upper estimate for $T_{2,n}(S_{\vec{q}})$.

Let I_{i_0} be such that $p, q = q_{i_0} \in I_{i_0}$. That is, $q_{i_0-1} < p < q_{i_0} = q$. Let $x^1, \dots, x^n \in S_{\vec{q}}$. Then, writing each x^j as $\sum_{i=1}^{\infty} x_i^j$ where $x_i^j \in \ell_{q_i}^{n_i}$, we have

$$\begin{aligned}
\operatorname*{Ave}_{\varepsilon_j=\pm1}\Big\|\sum_{j=1}^n \varepsilon_j x^j\Big\|_{S_{\vec{q}}}^2 &= \operatorname*{Ave}_{\varepsilon_j=\pm1}\Big\|\sum_{j=1}^n \varepsilon_j\Big(\sum_{i=1}^{\infty} x_i^j\Big)\Big\|_{S_{\vec{q}}}^2 = \operatorname*{Ave}_{\varepsilon_j=\pm1}\sum_{i=1}^{\infty}\Big\|\sum_{j=1}^n \varepsilon_j x_i^j\Big\|_{q_i}^2 \\
&\leq \sum_{i<i_0}\operatorname*{Ave}_{\varepsilon_j=\pm1}\Big\|\sum_{j=1}^n \varepsilon_j x_i^j\Big\|_{q_i}^2 + \sum_{i\geq i_0}\operatorname*{Ave}_{\varepsilon_j=\pm1}\Big\|\sum_{j=1}^n \varepsilon_j x_i^j\Big\|_{q_i}^2 \\
&\leq \sum_{i<i_0} T_2^2(\ell_{q_i}^{n_i})\sum_{j=1}^n \|x_i^j\|_{q_i}^2 + \sum_{i\geq i_0} T_{2,n}^2(\ell_{q_i}^{n_i})\sum_{j=1}^n \|x_i^j\|_{q_i}^2.
\end{aligned}$$

Using the estimates (4.2) for $i < i_0$ and (4.3) for $i \geq i_0$ sums, the last inequality is less than or equal to

$$\sum_{i<i_0} c^2 q_i n_i^{2/q_i-1}\sum_{j=1}^n \|x_i^j\|_{q_i}^2 + \sum_{i\geq i_0} c^2 q_i n^{2/q-1}\sum_{j=1}^n \|x_i^j\|_{q_i}^2,$$

which is, by (4.4), less than

$$2c^2 n^{2/q-1}\sum_{j=1}^n\sum_{i=1}^{\infty}\|x_i^j\|_{q_i}^2 = 2c^2 n^{2/q-1}\sum_{j=1}^n \|x^j\|_{S_{\vec{q}}}^2.$$

Thus, we have shown that $T_{2,n}(S_{\vec{q}}) \leq \sqrt{2}cn^{1/q-1/2}$. It follows that

$$\|T\|\|T^{-1}\| \geq \frac{n^{1/p-1/2}}{\sqrt{2}cn^{1/q-1/2}} = \frac{n^{1/p-1/q}}{\sqrt{2}c}.$$

□

5. THE COMPLEXITY OF THE UNIFORM HOMEOMORPHISM AND THE LOCAL EQUIVALENCE

In this section we prove the main theorems of our paper. In doing this we also define some natural equivalence relations and characterize their complexity. Some of the equivalence relations to be defined in this section have already been considered in [Ro]. For instance, Lemma 5.1, Definition 5.7, and the beginning of Theorem 5.8 can be found in [Ro]. For the sake of completeness we give all definitions and proofs in a self-contained manner.

For notational clarity we use the following convention in this section. Let X be a set. We use $\vec{x}$ to denote an element of X^ω, the set of the all infinite sequences of elements of X. The coordinates of $\vec{x}$ will be denoted by $x(n)$ for $n \in \omega$. Thus $\vec{x} = (x(n)) = (x(0), x(1), \dots)$. This is slightly different from previous sections, but it provides the most convenience for the arguments of this section.

Recall that the equivalence relation E_{ℓ_∞} (simply ℓ_∞ when there is no danger of confusion) is the equivalence relation on $\mathbb{R}^\omega$ defined by

$$\vec{x}\, E_{\ell_\infty}\, \vec{y} \iff \exists C\ \forall n\ |x(n)-y(n)|<C$$

for $\vec{x},\vec{y}\in\mathbb{R}^\omega$. We consider the following variation. Let B be the set of all infinite increasing sequences of positive real numbers without an upper bound. For any $\vec{b}=(b(0),b(1),\dots)\in B$, we denote by $E^{\vec{b}}_{\ell_\infty}$ the equivalence relation E_{ℓ_∞} restricted to the set $\prod_{n\in\omega}[0,b(n)]$.

Lemma 5.1. *For any $\vec{b}\in B$, $E_{\ell_\infty}\leq_B E^{\vec{b}}_{\ell_\infty}$.*

Proof. For each $n\in\omega$ let ρ_n be a linear map from $[-b(n),b(n)]$ onto $[0,b(n)]$. Define $\pi\colon\mathbb{R}^\omega\to\prod_{n\in\omega}[0,b(n)]$ by

$$\pi(\vec{x})(\langle i,j\rangle)=\begin{cases}\rho_j(x(i)), & \text{if } x(i)\in[-b(j),b(j)],\\ 0, & \text{if } x(i)<-b(j),\\ b(j), & \text{if } x(i)>b(j),\end{cases}$$

for all $i,j\in\omega$. Clearly $\pi(\vec{x})\in\prod_{n\in\omega}[0,b(n)]$. Note that if $\pi(\vec{x}_1)=\vec{y}_1$, $\pi(\vec{x}_2)=\vec{y}_2$, then

$$|y_1(n)-y_2(n)|\leq|x_1(i)-x_2(i)|$$

for all $n=\langle i,j\rangle\in\omega$. Thus $\vec{x}_1E_{\ell_\infty}\vec{x}_2$ implies $\pi(\vec{x}_1)E^{\vec{b}}_{\ell_\infty}\pi(\vec{x}_2)$. Suppose $\vec{x}_1$ is not E_{ℓ_∞}-equivalent to $\vec{x}_2$. Then for any $C>0$, there is an $i\in\omega$ such that $|x_1(i)-x_2(i)|>C$. Let j be a large enough integer such that $b(j)>\max\{|x_1(i)|,|x_2(i)|\}$. Let $n=\langle i,j\rangle$. Then $y_1(n)=\rho_j(x_1(i))$ and $y_2(n)=\rho_j(x_2(i))$, and so $|y_1(n)-y_2(n)|>C/2$. So, $\pi(\vec{x}_1)$ is not $E^{\vec{b}}_{\ell_\infty}$-equivalent to $\pi(\vec{x}_1)$. This shows that π is a reduction from E_{ℓ_∞} to $E^{\vec{b}}_{\ell_\infty}$. It is clear that π is a Borel function. □

We are now ready to prove our first main theorem.

Theorem 5.2. *The equivalence relation ℓ_∞ is Borel reducible to the uniform homeomorphism relation on either $\mathfrak{B}_b$ or $\mathfrak{B}$.*

Proof. By Lemma 5.1 it suffices to define a Borel reduction from $E^{\vec{b}}_{\ell_\infty}$ (for some $\vec{b}\in B$) to the uniform homeomorphism relation for Banach spaces with a basis. To construct the Banach spaces we use the proof of Theorem 4.1. For this fix $\vec{b}\in B\cap\omega^\omega$ with $b(0)>0$. For all $i\in\omega$ put

$$\delta_i=\frac{1}{b(i)2^i}\quad\text{and}\quad n_i=2^{\frac{1}{\delta_i}}=2^{b(i)2^i}.$$

Let I_i be a sequence of successive intervals in $(1,2)$ with $|I_i|=2^{-i-1}$. Assume $I_i=[l_i,r_i]$. Since $n_i^2\leq n_{i+1}$, equation (4.4) is satisfied. The sequences (n_i) and (I_i) will be used as in the proof Theorem 4.1. Let σ_i be the affine bijection between $[0,b(i)]$ and I_i. For $\vec{x}\in\prod_{n\in\omega}[0,b(n)]$, define $\rho(\vec{x})\in\mathbb{R}^\omega$ by

$\rho(\vec{x})(i) = \sigma_i(x(i))$. Finally, define $\pi(\vec{x}) = S_{\rho(\vec{x})}$. So, for all $\vec{x} \in \prod_{n\in\omega}[0, b(n)]$, $\pi(\vec{x})$ is a separable Banach space with a basis.

We show that π is the desired reduction. It is straightforward to check that π is Borel as a map from $\prod_{n\in\omega}[0, b(n)]$ to $\mathfrak{B}_b$. Granting that π is a reduction, then composed with the Borel map $\Phi : \mathfrak{B}_b \to \mathfrak{B}$, it would be a reduction to the uniform homeomorphism relation on $\mathfrak{B}$.

To verify that π is a reduction, consider $\vec{x}_1, \vec{x}_2 \in \prod_{n\in\omega}[0, b(n)]$. From Theorem 4.1 we have that $S_{\rho(\vec{x}_1)}$ and $S_{\rho(\vec{x}_2)}$ are uniformly homeomorphic iff

$$\exists C > 0\ \forall i \in \omega\ \ n_i^{\left|\frac{1}{\rho(\vec{x}_1)(i)} - \frac{1}{\rho(\vec{x}_1)(i)}\right|} < C.$$

By taking logarithm we get that this is equivalent to

$$\exists D > 0\ \forall i \in \omega\ \ \left|\frac{\log(n_i)}{\rho(\vec{x}_1)(i)} - \frac{\log(n_i)}{\rho(\vec{x}_2)(i)}\right| < D.$$

Using the definition of ρ we get that the inequality is equivalent to

$$\log(n_i)|\sigma_i(\vec{x}_1(i)) - \sigma_i(\vec{x}_2(i))|\frac{1}{\sigma_i(\vec{x}_1(i)) \cdot \sigma_i(\vec{x}_2(i))} < D.$$

Since $\sigma_i(\vec{x}_1(i)) \in (1, 2)$ and likewise for $\vec{x}_2$, the statement is thus equivalent to

$$\exists D > 0\ \forall i \in \omega\ \ \log(n_i)|\sigma_i(\vec{x}_1)(i) - \sigma_i(\vec{x}_2)(i)| < D.$$

By the linearity of σ_i, we in fact have

$$|\sigma_i(\vec{x}_1(i)) - \sigma_i(\vec{x}_2(i))| = \frac{|x_1(i) - x_2(i)|}{b(i) \cdot 2^{i+1}}.$$

Finally, our choice of (n_i) guarantees that

$$\frac{1}{2} \leq \left|\frac{\log(n_i)}{b(i) \cdot 2^i}\right| \leq 2.$$

Therefore, the statement is eventually equivalent to $\exists D > 0\ \forall i \in \omega\ |\vec{x}_1(i) - \vec{x}_2(i)| < D$, that is, $\vec{x}_1 E^{\vec{b}}_{\ell_\infty} \vec{x}_2$. □

Theorem 1.1 is now a direct corollary of the above proof. In particular we have the following corollary.

Theorem 5.3. *The equivalence relation* ℓ_∞ *is Borel reducible to the local equivalence on either* $\mathfrak{B}_b$ *or* $\mathfrak{B}$*, that is,* $\ell_\infty \leq_B\ \equiv_L$.

This gives a half of Theorem 1.2. Next we prove Theorem 1.2 by showing the reverse reduction. We will use the following concept and lemma.

Definition 5.4. For $X = (X, d)$ a Polish metric space, let F_X be the equivalence relation on X^ω defined by

$$\vec{x}\, F_X\, \vec{y} \iff \exists C > 0\ [\forall i\ \exists j\ d(x(i), y(j)) < C \wedge \forall i\ \exists j\ d(y(i), x(j)) < C].$$

Lemma 5.5. *For every Polish metric space* (X, d)*,* $F_X \leq_B \ell_\infty$.

Proof. Fix a 1-net $R = \{r_0, r_1, \dots\}$ in X. We define $\pi : X^\omega \to \mathbb{R}^\omega$ by

$$\pi(\vec{x})(i) = d(r_i, \{x(0), x(1), \dots\}).$$

It is easy to check that π is a Borel function. We verify that it is a reduction from F_X to E_{ℓ_∞}. Suppose $\vec{x}\, F_X\, \vec{y}$, and let $C > 0$ be a witness. For any $z \in X$, if $\delta = d(z, \{x(0), x(1), \dots\})$, then $d(z, \{y(0), y(1), \dots\}) \leq \delta + C$. So, $|d(z, \{x(0), x(1), \dots\}) - d(z, \{y(0), y(1), \dots\})| \leq C$. Thus, $\pi(\vec{x})\, E_{\ell_\infty}\, \pi(\vec{y})$.

Conversely, suppose $\vec{x}$ is not F_X-equivalent to $\vec{y}$. Let $C > 0$ be arbitrary. Then there is a k such that $d(x(k), \{y(0), y(1), \dots\}) > C$ or $d(y(k), \{x(0), x(1), \dots\}) > C$. Without loss of generality, assume the former. Let i be such that $d(x(k), r_i) < 1$. Then $\pi(\vec{x})(i) < 1$, but $\pi(\vec{y})(i) > C - 1$. So $\pi(\vec{x})$ is not E_{ℓ_∞}-equivalent to $\pi(\vec{y})$. □

Theorem 5.6. *The local equivalence on either $\mathfrak{B}$ or $\mathfrak{B}_b$ is Borel reducible to the equivalence relation ℓ_∞, that is, $\equiv_L \leq_B \ell_\infty$.*

Proof. Let $\mathcal{F}$ be the collection of finite dimensional Banach spaces (presented with bases). The following distance function is a separable metric on $\mathcal{F}$. If two spaces $(X, (x_1, \dots, x_n))$ and $(Y, (y_1, \dots, y_n))$ are both n-dimensional, let $\rho_n(X, Y) = \max\{\log(||T||), \log(||T^{-1}||)\}$, where $T : X \to Y$ is the linear isomorphism sending x_i to y_i. By truncating we may assume each $\rho_n \leq n$, and we may then put together the ρ_n to obtain a metric on $\mathcal{F}$ (if $\dim(X) \neq \dim(Y)$ we set $\rho(X, Y) = \dim(X) + \dim(Y)$). Let $(\bar{\mathcal{F}}, \rho)$ be the Polish space obtained by completing ρ.

First consider $\equiv_L$ as an equivalence relation on $\mathfrak{B}_b$. Given a separable Banach space X with basis (x_i), we define $\pi(X) \in \bar{\mathcal{F}}^\omega$ as follows. Let $\chi_1, \chi_2, \dots$ enumerate $(\mathbb{Q}^{<\omega})^{<\omega}$. Suppose $\chi_i = (\vec{q}_1, \dots, \vec{q}_k)$. Then let $\pi(X)(i)$ be the k-dimensional subspace of X with basis $(e_1, \dots, e_k)$, where $e_i = \sum \vec{q}_i(j) x_j$ for all $1 \leq i \leq k$. It is routine to check that π is a Borel function. Note that every finite-dimensional subspace of X is approximated arbitrarily closely in the ρ metric by a term of the sequence $\pi(X)$. It is then straightforward from the definition of the local equivalence that $X \equiv_L Y$ iff $\pi(X)\ F_{\bar{\mathcal{F}}}\ \pi(Y)$. Thus, π is a reduction from $\equiv_L$ to $E_{\bar{\mathcal{F}}}$. We are done by Lemma 5.5.

We modify the above argument to work for $\equiv_L$ as an equivalence relation on $\mathfrak{B}$. Let d be the metric on $C[0,1]$ given by the norm. Let $D \subseteq C[0,1]$ be countable dense. Fix an enumeration of $D^{<\omega} \times \omega$ as $(s_0, n_0), (s_1, n_1), \dots$. Fix a Borel function $\sigma\colon F(C[0,1]) \to C[0,1])$ such that $\sigma(F) \in F$ for all nonempty $F \in F(C[0,1])$ (see Theorem 12.13, [K]). For $x \in C[0,1]$, $F \in F(C[0,1])$, and $n \in \omega$, let

$$\sigma_n(x, F) = \begin{cases} \sigma\Big(\overline{F \cap \{u \in C[0,1] : d(x,u) < \frac{1}{n}\}}\Big), & \text{if } F \cap \{u \in C[0,1] : d(x,u) < \frac{1}{n}\} \neq \emptyset, \\ \sigma(F), & \text{otherwise.} \end{cases}$$

Given a separable Banach space $X \in \mathfrak{B}$, let $\pi(X)(i) \in \bar{\mathcal{F}}$ code the finite-dimensional subspace of X with basis $(\sigma_{n_i}(s_i(0), F), \dots, \sigma_{n_i}(s_i(|s_i|-1), F))$. Since D is dense, every finite-dimensional subspace of X is approximated arbitrarily closely by spaces of the form $\pi(F)(i)$. Thus, π is a Borel reduction from $\equiv_L$ to $E_{\bar{\mathcal{F}}}$. $\square$

Theorem 1.2 is immediate from Theorems 5.3 and 5.6. To summarize, we have shown that the local equivalence between separable Banach spaces has the same complexity as ℓ_∞, and the uniform homeomorphism relation is at least as complex as ℓ_∞. Thus we have obtained the sharpest result possible for the uniform classification by considering the local structures of Banach spaces alone.

The equivalence relation F_X we used in the above proof is sort of a generalization of the ℓ_∞ equivalence relation on the space of countable subsets of a Polish metric space equipped with the Hausdorff metric. In the remainder of this section we consider a full generalization of ℓ_∞ to arbitrary Polish metric spaces and characterize its complexity.

Definition 5.7. Let $X = (X, d)$ be a Polish metric space. The equivalence relation $E_{\ell_\infty(X)}$, or simply $\ell_\infty(X)$, on X^ω is defined as

$$\vec{x} E_{\ell_\infty(X)} \vec{y} \iff \exists C > 0\ \forall i \in \omega\ d(x(i), y(i)) < C.$$

We have the following dichotomy for the complexity of $\ell_\infty(X)$ in the Borel reducibility hierarchy for any Polish metric space X.

Theorem 5.8. *Let $X = (X, d)$ be a Polish metric space with d unbounded. Then $\ell_\infty(X)$ is Borel bireducible with either ℓ_∞ or E_1.*

Proof. We first reduce $\ell_\infty(X)$ to ℓ_∞. Fix a countable 1-*net* $R = \{r_0, r_1, \dots\}$ in X, that is, $R \subseteq X$ with $d(r_i, r_j) > 1$ for all $i \neq j$, and $\forall x \in X\ \exists i \in \omega\ d(x, r_i) \leq 1$. This can be done in any separable metric space. Define $\pi : X^\omega \to \mathbb{R}^\omega$ by

$$\pi(\vec{x})(\langle i, j\rangle) = d(x(i), r_j)$$

for any $i, j \in \omega$. Then π is continuous, in particular Borel. Let $\vec{x}, \vec{y} \in X^\omega$. If $\vec{x}\, E_{\ell_\infty(X)}\, \vec{y}$, then let $C > 0$ be such that $d(x(i), y(i)) < C$ for all $i \in \omega$. Then it follows that for any $j \in \omega$, $|d(x(i), r_j) - d(y(i), r_j)| < C$. So $\pi(\vec{x})\, E_{\ell_\infty}\, \pi(\vec{y})$. Conversely, if $\pi(\vec{x})\, E_{\ell_\infty}\, \pi(\vec{y})$, then for some $C > 0$ we have that $\forall i\ \forall j\ |d(x(i), r_j) - d(y(i), r_j)| < C$. It we take j so that $d(x(i), r_j) \leq 1$, then this implies that $d(x(i), y(i)) < C + 2$, and so $\vec{x}\, E_{\ell_\infty(X)}\, \vec{y}$.

Next we reduce E_1 to $\ell_\infty(X)$. Since $\mathbb{R}$ is Borel isomorphic to both 2^ω and ω^ω, we may work with E_1 defined on either $(2^\omega)^\omega$ or $(\omega^\omega)^\omega$, whichever is more convenient. Fix a sequence $(z_n) \in X^\omega$ with $\lim_n d(z_0, z_n) = \infty$. Define $\tau : (2^\omega)^\omega \to X^\omega$ by

$$\tau(\vec{x})(\langle i, j\rangle) = \begin{cases} z_i, & \text{if } x_i(j) = 1, \\ z_0, & \text{otherwise,} \end{cases}$$

for all $i, j \in \omega$. Again τ is continuous, hence Borel. Given $\vec{x}, \vec{x}' \in (2^\omega)^\omega$ and $n \in \omega$, we have that $\forall k \geq n\ x_k = x'_k$ iff $\tau(\vec{x})$, $\tau(\vec{x}')$ only disagree where they take values in $\{z_0, \dots, z_{n-1}\}$. This implies that $\vec{x} E_1 \vec{x}'$ iff $\tau(\vec{x}) E_{\ell_\infty(X)} \tau(\vec{x}')$.

We have shown so far that $E_1 \leq_B \ell_\infty(X) \leq_B \ell_\infty$. If Y is a 1-net in X, then clearly $\ell_\infty(X)$ is bireducible with $\ell_\infty(Y)$. So, without loss of generality we may assume that X is countable. For every positive C, let $\sim_C$ be the equivalence relation on X which is the transitive closure of the relation $\{(x,y)\colon d(x,y) < C\}$. We call the $\sim_C$-equivalence classes the *C-components* of X. We consider now two cases.

Case I: For all C there is a bound K_C on the diameter of the C-components.

In this case we reduce $\ell_\infty(X)$ to E_1. For each positive integer n, let $A^n_0, A^n_1, \dots$ enumerate (with repetition) the n-components of X. Given $\vec{x} = (x_0, x_1, \dots) \in X^\omega$, define $\vec{y} = \pi(\vec{x}) \in (\omega^\omega)^\omega$ by: $y_n(m) = j$ iff $x_m \in A^n_j$. Suppose first that $\vec{x}\, E_{\ell_\infty(X)}\, \vec{x}'$, say $\forall n\ d(x_n, x'_n) \leq N$. Then, for all $n \geq N$ we have that for all m, x_m and x'_m lie in the same n-component, since any two points in two distinct n-components have d distance greater than n. This shows that $y_n = y'_n$ for all $n \geq N$, and so $\vec{y}\, E_1\, \vec{y}'$. Conversely, suppose $\forall n \geq N\ y_n = y'_n$. So, for all $n \geq N$ and all m, x_m and x'_m lie in the same n-component. In particular, x_m and x'_m lie in the same N-component for all m, and so $d(x_m, x'_m) \leq K_m$ for all m, that is, $\vec{x}\, E_{\ell_\infty(X)}\, \vec{x}'$.

Case II: For some C and every K, there is a C-component of diameter greater than K.

In this case we reduce ℓ_∞ to $\ell_\infty(X)$. Fix C as in the case hypothesis. By a *C-path* we mean a finite sequence of points $y_0, y_1, \dots, y_n$ from X such that $d(y_i, y_{i+1}) < C$ for all i. Note that all the points of a C-path lie in the same C-component of X. Let $p_0, p_1, \dots$ enumerate all of the C-paths in X. If $p = (y_0, \dots, y_n)$ is a C-path and $i \in \omega$, let $p(i) = y_i$ if $i \leq n$ and otherwise let $p(i) = y_n$. Clearly $d(p(i), p(j)) \leq C|i-j|$ for any C-path p and any $i, j \in \omega$. Given $x \in \omega^\omega$, define $\vec{y} = \pi(x) \in X^\omega$ by $y_{\langle i,j\rangle} = p_i(x(j))$. If $\forall m\ |x(m) - x'(m)| \leq N$, then $\forall m\ d(y_m, y'_m) \leq CN$ from the above observation. Suppose then that x is not E_{ℓ_∞}-equivalent to x'. Given k, let $A \subseteq X$ be a C-component with diameter greater than k. Let $z, w \in A$ with $d(z,w) > k$. Let p be a C-path from z to w. Say $p = (z = z_0, z_1, \dots, w = z_n)$. Let i be such that $|x(i) - x'(i)| > n$. From p we can easily obtain a C-path q such that $q(x(i)) = z_0$ and $q(x'(i)) = z_n$ (have the path q start at z_0, remain at z_0 for an appropriate number of steps, then follow p, and then remain at z_n). Say $q = q_j$. Then $y_{\langle j,i\rangle} = q(x(i)) = z_0$ and $y'_{\langle j,i\rangle} = q(x'(i)) = z_n$. Thus $d(y_{\langle j,i\rangle}, y'_{\langle j,i\rangle}) \geq k$. Since this is true for all k, we have that $\vec{y}$ is not $E_{\ell_\infty(X)}$-equivalent to $\vec{y}'$. □

6. Some special classes of separable Banach spaces

In this section we generalize the construction in Section 4 to obtain some classes of separable Banach spaces. For each of these classes it turns out that the isomorphism, the uniform homeomorphism, and the local equivalence relations on it coincide. We also obtain some characterizations for the possible complexity of these equivalence relations.

We will use the following equivalence relation on 2^ω and a characterization of its possible complexity.

Definition 6.1. For any sequence $\vec{t} = (t_i) \in \mathbb{R}^\omega$ with $t_i \geq 0$ for all $i \in \omega$, let $E_{\vec{t}}$ be the equivalence relation on 2^ω defined by

$$x \, E_{\vec{t}} \, y \iff \sup_i \, (t_i \cdot |x(i) - y(i)|) < \infty.$$

Theorem 6.2. *For any $\vec{t} \in \mathbb{R}^\omega$ with $t_i \geq 0$ for all $i \in \omega$, $E_{\vec{t}}$ is either smooth, Borel bireducible with E_0, or Borel bireducible with E_1.*

Proof. If (t_i) is bounded, then $E_{\vec{t}}$ is trivial, and in particular smooth. So we assume $\vec{t}$ is unbounded. We inductively define a finite or infinite sequence $n_0 < n_1 < \dots$ of natural numbers as follows. Let n_0 be the least $n \in \omega$, if one exists, such that $\{i \colon t_i \leq n\}$ is infinite. Suppose n_k is defined, then let n_{k+1} be the least $n > n_k$, if one exists, such that $\{i \colon n_k < t_i \leq n\}$ is infinite. If n_k is defined, we also let $A_k = \{i \colon n_{k-1} < t_i \leq n_k\}$.

First assume that n_k is defined for all $k \in \omega$. Thus, A_k is defined for all k and the A_k form a partition of ω. Note that each A_k is infinite by definition. Let e^i_k, $i \in \omega$, enumerate A_k. Define $f : 2^\omega \to (2^\omega)^\omega$ by $f(x)_k(i) = x(e^i_k)$. Clearly $x_1 \, E_{\vec{t}} \, x_2$ iff the sequences of reals coded by $f(x_1)$ and $f(x_2)$ are eventually the same, that is, $f(x_1) E_1 f(x_2)$. Thus, f is a Borel reduction of $E_{\vec{t}}$ to E_1. In fact, f is a bijection between 2^ω and $(2^\omega)^\omega$, so its inverse gives a reduction from E_1 to $E_{\vec{t}}$.

Suppose next that n_0 is not defined. In this case $t_i \to \infty$. Then in fact $x \, E_{\vec{t}} \, y$ iff $x \, E_0 \, y$, that is, the identity map is a reduction from $E_{\vec{t}}$ to E_0. Since the identity map is again a bijection, we have that $E_{\vec{t}}$ is Borel bireducible with E_0.

Finally, suppose that $n_0 < \cdots < n_\ell$ are defined, while $n_{\ell+1}$ is not. Since (t_i) is unbounded, we must have that $\omega - \bigcup_{k \leq \ell} A_k$ is infinite. Let e^i_k, $i \in \omega$, enumerate A_k for $k \leq \ell$, and let $e^i_{\ell+1}$, $i \in \omega$, enumerate $\omega - \bigcup_{k \leq \ell} A_k$. Define $g : 2^\omega \to 2^\omega$ by $g(x)(i) = x(e^i_{\ell+1})$. Clearly f is a Borel reduction of $E_{\vec{t}}$ to E_0. For the other direction, define $h : 2^\omega \to 2^\omega$ by

$$h(y)(j) = \begin{cases} y(i),, & \text{if } j = e^i_{\ell+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Easily h is a reduction of E_0 to $E_{\vec{t}}$. □

The above proof can be simplified in view of known facts about E_1 and E_0 (see Section 2). In fact, if $E \leq_B E_1$ then E is either smooth or Borel bireducible with either E_0 or E_1 by the dichotomy theorems of [HaKL] and [KL]. However, we gave the full proof here since it is self-contained and gives some information about the combinatorial structure of the equivalence relation $E_{\vec{t}}$. This will happen again for the proof of Theorem 6.5 below.

As in Section 4 we consider sequences $\vec{p} = (p_i)$, $\vec{q} = (q_i) \in \mathbb{R}^\omega$, and $\vec{n} = (n_i) \in \omega^\omega$ such that

$$1 < p_i < q_i < p_{i+1} < 2,\ n_i > 0 \text{ and } n_{i+1}^{\frac{1}{q_{i+1}}} > n_i^{\frac{1}{p_i}}. \tag{6.1}$$

Let $\mathcal{B}_{\vec{p},\vec{q},\vec{n}}$ be the collection of Banach spaces of the form

$$X = \big(\sum_{i=0}^{\infty} \oplus \ell_{r_i}^{n_i}\big)_2,$$

where $r_i \in \{p_i, q_i\}$. $\mathcal{B}_{\vec{p},\vec{q},\vec{n}}$ can be viewed as a closed subspace of $\mathfrak{B}_b$. To see this, first code the elements of $\mathcal{B}_{\vec{p},\vec{q},\vec{n}}$ by elements of 2^ω in the natural manner (i.e., $x(i)$ determines whether to use ℓ_{p_i} or ℓ_{q_i}). By using a fixed bijection between $\omega \times \omega$ and ω, we fix an order of enumeration of the basis elements for all the spaces in $\mathcal{B}_{\vec{p},\vec{q},\vec{n}}$. This induces a map f from 2^ω to $\mathfrak{B}_b \subseteq \mathbb{R}^\omega$ which is easily seen to be continuous. Then $f(2^\omega)$ is a closed subset of $\mathfrak{B}_b$ which represents the set of spaces in $\mathcal{B}_{\vec{p},\vec{q},\vec{n}}$. Clearly each $\mathcal{B}_{\vec{p},\vec{q},\vec{n}}$ contains continuum many elements.

Theorem 6.3. *For any $\vec{p}$, $\vec{q}$, $\vec{n}$ satisfying* (6.1) *above, the uniform homeomorphism relation on $\mathcal{B}_{\vec{p},\vec{q},\vec{n}}$ is either smooth, Borel bireducible to E_0, or Borel bireducible to E_1.*

Proof. Consider the sequence of numbers $t_i = n_i^{\frac{1}{p_i} - \frac{1}{q_i}}$. First suppose that the sequence (t_i) is bounded. In this case, all of the spaces $\mathcal{B}_{\vec{p},\vec{q},\vec{n}}$ are isomorphic, so the uniform homeomorphism relation on $\mathcal{B}_{\vec{p},\vec{q},\vec{n}}$ is trivial.

Suppose next that (t_i) is unbounded. For each $X \in \mathcal{B}_{\vec{p},\vec{q},\vec{n}}$, let $z(X) \in 2^\omega$ be the real z such that $z(i) = 0$ if X involves $\ell_{p_i}^{n_i}$ and $z(i) = 1$ if X involves $\ell_{q_i}^{n_i}$.

From the proof of Theorem 4.1 we have that for $X, Y \in \mathcal{B}_{\vec{p},\vec{q},\vec{n}}$, X is uniformly homeomorphic to Y iff $\sup_i(t_i \cdot |z(X)(i) - z(Y)(i)|) < \infty$, that is, $z(X)\ E_{\vec{t}}\, z(Y)$. Therefore we are done by Theorem 6.2. □

We now extend Theorem 6.3 to some even larger classes of separable Banach spaces. Again we define and study some new equivalence relations.

Definition 6.4. For any sequence $\vec{B} = (B_i)$ where each B_i is a finite subset of $\mathbb{R}$, let $E_{\vec{B}}$ denote the equivalence relation ℓ_∞ restricted on $\prod_{i\in\omega} B_i$.

For $\vec{B} = (B_i)$ as in the above definition let $b_i = \sup\{|a| : a \in B_i\}$. Then $E_{\vec{B}}$ is also $E_{\ell_\infty}^{\vec{b}}$ restricted to $\prod_{i\in\omega} B_i$. Thus $E_{\vec{B}} \leq_B E_{\ell_\infty}^{\vec{b}} \leq_B \ell_\infty$.

Theorem 6.5. *Let $\vec{B} = (B_i)$ where each B_i is a finite subset of $\mathbb{R}$. Then $E_{\vec{B}}$ is either smooth, Borel bireducible with E_0, Borel bireducible with E_1, or Borel bireducible with ℓ_∞.*

Proof. By translating each B_i we may assume that each B_i consists of non-negative real numbers and contains 0 as its least element. Let $b_i = \max B_i$. If (b_i) is bounded, then E is a trivial equivalence relation, and so is smooth.

So, we assume (b_i) is unbounded. Also, we may assume that $B_i \subseteq \omega$, for we may replace B_i by $\{\lfloor a \rfloor : a \in B_i\}$.

For i, $n \in \omega$, let F^i_n denote the finite equivalence relation on B_i given by the transitive closure of the relation

$$xR^i_n y \iff |x-y| \le n.$$

For each i, n, let $a^i_n(0), \dots, a^i_n(k)$ enumerate the F^i_n classes of B_i in increasing order (i.e., $\max(a^i_n(l)) < \min(a^i_n(l+1))$). Here $k = k(i,n)$ depends on i and n.

First consider the case where for some n, there is no bound on the size of the F^i_n equivalence classes. That is, $\forall b\ \exists i\ \exists l\ |a^i_n(l)| > b$. Fix such an n. Let $i_0 < i_1 < \dots$ be a subsequence and $l_0, l_1, \dots$ a sequence such that $|a^{i_m}_n(l_m)| > m$. We know that $E_{\ell_\infty} \le E_{\vec{C}}$ where $C_i = \{0, 1, \dots, i\}$. So, it suffices to show in this case that $E_{\vec{C}} \le E_{\vec{B}}$, as it then gives that $E_{\vec{B}}$ is Borel bireducible with ℓ_∞. Let $Z = \prod C_i$ and $X = \prod B_i$. Define $\pi\colon Z \to X$ by

$$\pi(z)(i) = \left\{ \begin{array}{ll} 0, & \text{if } i \notin \{i_0, i_1, \dots\}, \\ \text{the } z(m)\text{-th element of } a^i_n(l_m), & \text{if } i = i_m. \end{array} \right.$$

Then for all $x, y \in Z$ we have $|x(m) - y(m)| \le |\pi(x)(i_m) - \pi(y)(i_m)| \le n|x(m) - y(m)|$. It follows that π is a Borel reduction from $E_{\vec{C}}$ to $E_{\vec{B}}$.

Next consider the case where for each n there is a bound K_n on the size of the F^i_n equivalence classes, that is, $\forall i\ \forall l\ |a^i_n(l)| < K_n$. We first show in this case that $E_{\vec{B}} \le_B E_1$. We define a map τ from $X = \prod B_i$ to $(\omega^\omega)^\omega$ as follows. For $x \in X$ let $\tau(x)(n) \in \omega^\omega = y_n$ be the real such that $y_n(i) =$ the unique l such that $x(i) \in a^i_n(l)$. Consider $x, y \in X$. If $xE_{\vec{B}}\, y$, then for some $C > 0$ we have $|x(i) - y(i)| < C$ for all i. Let n be such that $n > C$. Then for all i we must have that $x(i)$, $y(i)$ lie in the same class of F^i_n, since any two points in distinct F^i_n class are at least n apart. This shows that $\tau(x)(m) = \tau(y)(m)$ for all $m \ge n$. That is, $\tau(x) E_1(\omega)\, \tau(y)$, where $E_1(\omega)$ refers to the E_1 (eventual agreement) relation on $(\omega^\omega)^\omega$. Conversely, suppose $\tau(x) E_1(\omega)\, \tau(y)$. Fix n so that for all $m \ge n$, $\tau(x)(m) = \tau(y)(m)$. Then for all i we have that $|x(i) - y(i)| \le nK_n$, since any two points in the same F^i_n equivalence class are at most nK_n apart. Thus, $xE_{\vec{B}}\, y$. Thus, τ is a Borel reduction of $E_{\vec{B}}$ to $E_1(\omega)$. However, it is easy to see that $E_1(\omega) \le_B E_1$, so $E_{\vec{B}} \le_B E_1$ in this case.

We next consider subcases. First assume that for every n and every M there is a D_n such that for infinitely many i we have that $M < g^i_n < D_n$, where g^i_n is the minimum distance between distinct F^i_n classes (and $= 0$ if there is only one F^i_n class). We may therefore get a sequence $k_0 < k_1 < \cdots$ such that for all n, there are infinitely many i such that $k_n < g^i_n \le k_{n+1}$. For each n, let $A_n = \{i\colon k_n < g^i_n < k_{n+1}\}$. Note that the A_n are pairwise disjoint (this follows from the fact that $g^i_m \le g^i_n$ if $m > n$). for each n and each $i \in A_n$, let $l = l(i,n)$ and $l' = l'(i,n)$ be such that the distance between $a^i_n(l)$ and $a^i_n(l')$ is between k_n and k_{n+1}. Define $\varphi\colon (2^\omega)^\omega \to X$ as follows. Given $y = (y_0, y_1, \dots) \in (2^\omega)^\omega$, let $\varphi(y) = x \in X$ where $x(i) = 0$

if $i \notin \bigcup_n A_n$, and for $i \in A_n$, say if i is the j^{th} element of A_n, then $x(i)$ is the least element of $a_n^i(l)$ if $y_n(j) = 0$ and the least element of $a_n^i(l')$ if $y_n(j) = 1$. Note that for any x, x' in the range of φ, we always have that for all $i \in A_n$ that $|x(i) - x'(i)| \leq k_{n+1} + nK_n$. Also, if $y_n \neq y_n'$, then for some $i \in A_n$ we have that $|x(i) - x'(i)| \geq k_n$. It follows that φ is a Borel reduction of E_1 to $E_{\vec{B}}$. Thus, $E_{\vec{B}}$ is Borel bireducible with E_1.

Finally assume that for some n we have that

$$\forall C > 0 \; \exists i_C \; \forall i \geq i_C \; g_n^i > C.$$

Define $\psi\colon X \to \omega^\omega$ by $\psi(x)(i) =$ the unique l such that $x(i) \in a_n^i(l)$. If $x E_{\vec{B}}\, y$, then we must have $\psi(x) E_0 \, \psi(y)$ as g_n^i tends to infinity with i. Conversely, if $\psi(x) E_0 \, \psi(y)$, then $x E_{\vec{B}}\, y$ since $|x(i) - y(i)| \leq nK_n$ for all i. So, $E_{\vec{B}} \leq E_0$ in this case. More generally, if we assume that for some n and some M that

$$\forall C > 0 \; \exists i_C \; \forall i \geq i_C \; (g_n^i > C \vee g_n^i < M),$$

then the same conclusion follows. This is because on the set A of i such that $g_n^i < M$ we have that B_i consists of a single F_M^i class, and thus $|B_i| \leq K_M$ for these i. Thus, $\max B_i \leq MK_M$ for such i, and so we proceed as before to define ψ, except now we use only those $i \notin A$ (i.e., set $\psi(x)(i) = 0$ for $i \notin A$). It is also easy to reduce E_0 to $E_{\vec{B}}$ in this case and so $E_{\vec{B}}$ is Borel bireducible with E_0. The argument is an easier variation of that given in the preceding paragraph. $\square$

To define our generalized classes of Banach spaces, we again fix a sequence of successive intervals $I_i = [l_i, r_i]$ with $l_{i+1} > r_i$, and integers $\vec{n} = (n_i)$ as in Theorem 4.1. Once again, we assume that

$$n_{i+1}^{\frac{1}{r_{i+1}}} \geq n_i^{\frac{1}{l_i}}.$$

For each i, let $S_i \subseteq [l_i, r_i]$ be a finite set.

Definition 6.6. For I_i, n_i, and S_i as above, let $\mathcal{B}_{\vec{S},,\vec{n}}$ be the collection of separable Banach spaces of the form $X = \left(\sum_{i=1}^\infty \oplus \ell_{r_i}^{n_i}\right)_2$, where $r_i \in S_i$. Let $E_{\vec{S},\vec{n}}$ denote the uniform homeomorphism relation on the collection $\mathcal{B}_{\vec{S},,\vec{n}}$.

We note that $\mathcal{B}_{\vec{S},,\vec{n}}$ can be regarded as a closed subspace of $\mathfrak{B}_b$. Alternatively, we may regard $\mathcal{B}_{\vec{S},,\vec{n}}$ as the space $\prod_i S_i$ (S_i having the discrete topology) which is homeomorphic to 2^ω. These two topologies give the same Borel structure on $\mathcal{B}_{\vec{S},,\vec{n}}$.

Theorem 6.7. *For any $\vec{I}$, $\vec{S}$, $\vec{n}$ as above, $E_{\vec{S},\vec{n}}$ is either smooth, Borel bireducible with E_0, Borel bireducible with E_1, or Borel bireducible with ℓ_∞.*

Proof. Suppose $X, Y \in \mathcal{B}_{\vec{S},,\vec{n}}$, say X corresponds to the sequence (p_i) (where $p_i \in S_i$), and Y corresponds to (q_i). Again the proof of Theorem 4.1 shows that $X E_{\vec{S},\vec{n}} Y$ iff $\left(n_i^{|\frac{1}{p_i} - \frac{1}{q_i}|}\right)$ is bounded. Define $\pi\colon \mathcal{B}_{\vec{S},,\vec{n}} \to \mathbb{R}^\omega$ as follows. If

X corresponds to the sequence (p_i), then let $\pi(X)(i) = \frac{1}{p_i}\log(n_i)$. Note that all of the $\pi(X)(i)$ take values in the finite set $B_i := \{\frac{1}{p_i}\log(n_i) \colon p_i \in S_i\}$. We then have that $XE_{\vec{S},\vec{n}}Y$ iff $\pi(X)E_{\vec{B}}\pi(Y)$. Moreover, Moreover, π is a bijection between $\mathcal{B}_{\vec{S},,\vec{n}}$ and $\prod B_i$. Thus $E_{\vec{S},\vec{n}}$ is Borel bireducible with $E_{\vec{B}}$. We are done by Theorem 6.5. $\square$

7. Nonisomorphic uniformly homeomorphic Banach spaces

Fix a countable dense set $D \subseteq (2,3)$. If A is a countable subset of $(2,3)\setminus D$, then we associate to A the separable Banach space

$$X_A = \left(\sum_{p\in D} \oplus \ell_p\right)_{c_0} \bigoplus \left(\sum_{q\in A} \oplus \ell_q\right)_{c_0} .$$

Since $(2,3)\setminus D$ is Borel bijectable with $\mathbb{R}$, to prove Theorem 1.3 it suffices to show that X_A is uniformly homeomorphic to X_B for any countable $A, B \subseteq (2,3)\setminus D$, and X_A is not isomorphic to X_B for $A \neq B$. The following well known lemma, which we sketch a proof for convenience, verifies the second requirement.

Lemma 7.1. *If $A \subseteq (2,3)$ is countable and $q \notin A$, then ℓ_q is not isomorphic to a subspace of $\left(\sum_{p\in A} \oplus \ell_p\right)_{c_0}$.*

Proof. Let $A = \{p_n : n \in \mathbb{N}\}$. For $k \in \mathbb{N}$, denote by P_k the natural projection onto $\left(\sum_{n=1}^{k} \oplus \ell_{p_n}\right)_{c_0}$. Suppose there exists a $X \subset (\sum_{n=1}^{\infty} \oplus \ell_{p_n})_{c_0}$ which is isomorphic to ℓ_q. Consider two mutually exclusive cases.

(i) Suppose that there exist $\varepsilon > 0$ and $k \in \mathbb{N}$ such that for all $x \in X$, $\|P_k x\| \geq \varepsilon\|x\|$. Put $X' = P_k(X)$. Then $T : X \to X'$ defined by $Tx = P_k(x)$, for all $x \in X$, is an isomorphism with $\|T^{-1}\| \leq 1/\varepsilon$. That is, ℓ_q is isomorphic to a subspace of the finite direct sum $\left(\sum_{n=1}^{k} \oplus \ell_{p_n}\right)_{c_0}$, which is impossible unless $q = p_n$ for some $1 \leq n \leq k$.

(ii) Suppose that for all $\varepsilon > 0$ and all $k \in \mathbb{N}$ there exists normalized $x \in X$ with $\|P_k x\| < \varepsilon\|x\|$. Let $(\varepsilon_i) \searrow 0$ such that $\sum_i \varepsilon_i < 1/4$. Construct inductively a sequence of normalized $(x_i) \in X$ and $0 < k_1 < k_2 < k_3 \ldots$ such that $\|x_i - P_{k_i}x_i\| \leq \varepsilon_i$ and $\|P_{k_i}x_{i+1}\| < \varepsilon_i$, and put $x_i' = (P_{k_i} - P_{k_{i-1}})x_i$. Then $(x_i')_{i=1}^{\infty}$ is a sequence of disjointly supported vectors thus equivalent to the unit vector basis of c_0. Since $\sum_{i=1}^{\infty} \|x_i - x_i'\| \leq \sum_i(\varepsilon_i + \varepsilon_{i-1}) < 1/2$, this implies that $(x_i) \subset X$ is equivalent to c_0 basis, a contradiction. $\square$

Theorem 7.2. *Let $D \subset (2,3)$ be dense and $A \subset (2,3)\setminus D$ be countable. Then $X = \big(\sum_{p\in D} \oplus \ell_p\big)_{c_0}$ is uniformly homeomorphic to $X_A = \big(\sum_{p\in D} \oplus \ell_p\big)_{c_0} \oplus \big(\sum_{q\in A} \oplus \ell_q\big)_{c_0}$.*

Remark 7.3. The proof is a slight generalization of Theorem 10.28 in [BL]. The idea of the proof is due to Ribe [Ri2] who proved it in a special case. This was later extended by Aharoni and Lindenstrauss [AL] to a more general setting (see [Ben] for a nice exposition). We will reproduce the main steps of the proof following [BL] with the necessary modifications, and also present an additional step (Lemma 7.4) clarifying an obscure point there.

Proof. Recall that for $x = (x_i) \in \ell_p$, the Mazur map $\varphi_{p,q} : \ell_p \to \ell_q$ is defined by

$$\varphi_{p,q}(x) = \|x\|_p^{1-\frac{p}{q}} (\mathrm{sign}(x_i)|x_i|^{\frac{p}{q}})_i.$$

φ is positively homogeneous and, for each $K > 0$ it is a uniform homeomorphism of K-ball in ℓ_p onto the K-ball in ℓ_q. Moreover, for every M the family $\{\varphi_{p,q} : 1 \leq p, q \leq M\}$ is a family of equi-uniform homeomorphisms where each $\varphi_{p,q}$ is restricted to the ball of radius $\exp(1/|p-q|)$ (see Proposition 9.2, [BL]).

Let (q_j) be an enumeration of A. Since D is dense, there exist disjoint infinite subsets $I_j = \{\langle j, n\rangle : n \in \mathbb{N}\} \subset \mathbb{N}$, $j = 1, 2 \ldots$, such that $p_{\langle j,n\rangle} \to q_j$ for each j. To simplify the notation, we write $\varphi_{j,n}$ for the Mazur map $\varphi_{p_{\langle j,n\rangle}, q_j} : \ell_{p_{\langle j,n\rangle}} \to \ell_{q_j}$. By passing to subsequences of I_j's if necessary, we can and will assume that the family $\{\varphi_{j,n}, (\varphi_{j,n})^{-1} : j, n \in \mathbb{N}\}$ of maps where each is restricted to the 2^n-balls of its domain is equi-uniformly continuous.

In the next step we solely work on copies of ℓ_{q_j}'s. The goal is to construct continuous paths of homeomorphisms between two particular invertible operators S_0^j and S_1^j described below in a 'uniform' manner. For a fixed j, this follows from the fact that the general linear group of invertible operators on ℓ_q is contractible (cf. e.g., [Mi]). Since we require the paths to be independent of j's, we give them explicitly.

Lemma 7.4. *There exists a continuous path $\tau \to V_\tau$, $0 \leq \tau \leq 1/2$ of invertible operators on $(\ell_q \oplus \ell_q \oplus \ell_q)_q$ such that V_0 is the identity and $V_{1/2}(u, v, w) = (u, w, v)$, for all $(u, v, w) \in (\ell_q \oplus \ell_q \oplus \ell_q)_q$. Moreover, $\|V_\tau\| \leq 2$, and the path is independent of $1 \leq q < \infty$ in the sense that the matrix representation of V_τ with respect to the decomposition $\ell_q(\ell_q)$ does not depend on q.*

Proof. Consider an isomorphism $D : \ell_q \to \ell_q(\ell_q)$. D induces an isomorphism $(\ell_q \oplus \ell_q \oplus \ell_q)_q \to (\ell_q \oplus \ell_q)_q \oplus (\ell_q \oplus \ell_q \oplus \ldots)_q$ mapping (u, v, w) to $(v, w, (Du)_1, (Du)_2, \ldots)$. Regarding the latter as a sequence of scalars and composing with obvious isometries, the operator $V_{1/2}$ can be written as a block diagonal matrix of the form $V_{1/2} = J \oplus I \oplus I \oplus \ldots$ where I is the 2×2 identity matrix and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For $0 \leq \tau \leq 1/4$, consider the path V_τ of invertible operators defined by $V_\tau = A_\tau \oplus A_\tau \oplus A_\tau \oplus \ldots$ where $A_\tau = \begin{pmatrix} B_\tau & C_\tau \\ -C_\tau & B_\tau \end{pmatrix}$, and

$$B_\tau = \begin{pmatrix} \cos^2 2\pi\tau & \sin^2 2\pi\tau \\ \sin^2 2\pi\tau & \cos^2 2\pi\tau \end{pmatrix}, \quad C_\tau = \begin{pmatrix} -\cos 2\pi\tau \sin 2\pi\tau & \cos 2\pi\tau \sin 2\pi\tau \\ \cos 2\pi\tau \sin 2\pi\tau & -\cos 2\pi\tau \sin 2\pi\tau \end{pmatrix}.$$

Thus, the path connects the identity to $V_{1/4} = J \oplus J \oplus \ldots$.

For $1/4 \leq \tau \leq 1/2$, we continue the path by $V_\tau = J \oplus A_\tau \oplus A_\tau \oplus \ldots$. Thus $V_{1/2} = J \oplus I \oplus I \oplus \ldots$, as desired. Note that since $A_{1/4} = J \oplus J$, two definitions of $V_{1/4}$ coincide and therefore it is well defined. Clearly, the paths are independent of $1 \leq q < \infty$, and an easy computation shows that $\|V_\tau\| \leq 2$ for all $0 \leq \tau \leq 1/2$. ☐

Now for each j let $T_j : (\ell_{q_j} \oplus \ell_{q_j})_{q_j} \to \ell_{q_j}$ be a linear isometry, and consider the following isometries from $(\ell_{q_j} \oplus \ell_{q_j} \oplus \ell_{q_j})_{q_j}$ onto $(\ell_{q_j} \oplus \ell_{q_j})_{q_j}$ induced by T_j's:

$$\begin{aligned} S_0^j(u,v,w) &= (T_j(u,v),w) \text{ and} \\ S_1^j(u,v,w) &= (v,T_j(u,w)) \text{ for } (u,v,w) \in (\ell_{q_j} \oplus \ell_{q_j} \oplus \ell_{q_j})_{q_j}. \end{aligned}$$

Lemma 7.5. *For all j and $0 \leq \tau \leq 1$, there is a homogeneous norm-preserving homeomorphism $h_\tau^j : (\ell_{q_j} \oplus \ell_{q_j} \oplus \ell_{q_j})_{q_j} \to (\ell_{q_j} \oplus \ell_{q_j})_{q_j}$ such that $h_0^j = S_0^j$ and $h_1^j = S_1^j$, and such that there is a constant K (independent of j) for which*

$$\|h_\tau^j(x) - h_\eta^j(y)\| \leq K\big(\|x-y\| + |\tau - \eta| \max(\|x\|,\|y\|)\big)$$

and similarly for their inverses.

Proof. For all j and $0 \leq \tau \leq 1/2$, let V_τ^j be given by Lemma 7.4 for q_j. We define S_τ^j for all j and $0 \leq \tau \leq 1$ as follows. For $0 \leq \tau \leq 1/2$, define $S_\tau^j(u,v,w) = (T_j(u_\tau,v_\tau),w_\tau)$, where $(u_\tau,v_\tau,w_\tau) = V_\tau^j(u,v,w)$. For $1/2 \leq \tau \leq 1$, put $S_\tau^j = U_\tau^j S_{1/2}^j$ where U_τ^j is a path of invertible operators on $(\ell_{q_j} \oplus \ell_{q_j})_{q_j}$ connecting the identity to the operator $(u,v) \to (v,u)$. To get the path U_τ^j, start with the isomorphism $E^j : (\ell_{q_j} \oplus \ell_{q_j})_{q_j} \to (\ell_{q_j} \oplus \ell_{q_j} \oplus \ldots)_{q_j}$ defined by $E^j(u,v) = ((D^ju)_1,(D^jv)_1,(D^ju)_2,(D^jv)_2,\ldots)$ where $D^j : \ell_{q_j} \to \ell_{q_j}(\ell_{q_j})$ is an isomorphism. Then put $U_\tau^j = (E^j)^{-1}\tilde{V}_\tau^j E^j$ where $\tilde{V}_\tau^j = V_{(2\tau-1)/4}^j$. Note that the norm of S_τ^j's and their inverses are uniformly bounded, and it is clear from the formulas that S_τ^j's are Lipschitz in τ. Finally, putting $h_\tau^j(x) = \|x\|S_\tau^j(x)/\|S_\tau^j(x)\|$ yields the desired norm-preserving maps. Note that the inverses of the normalized maps have the same form, that is, $(h_\tau^j)^{-1}(y) = \|y\|(S_\tau^j)^{-1}(y)/\|(S_\tau^j)^{-1}(y)\|$. ☐

The desired uniform homeomorphism from X_A onto X will be defined by $\phi(x) = g_{\|x\|}(x)$ where $g_t(x) = \|x\|\tilde{g}_t(x)/\|\tilde{g}_t(x)\|$, $t > 0$ and $\tilde{g}_t$ is defined below.

Every $x \in X_A$ has a unique representation of the form $\sum_j u_j + \sum_i x_i$ where $u_j \in \ell_{q_j}$, $q_j \in A$ and $x_i \in \ell_{p_i}$, $p_i \in D$. For notational convenience we split the second sum and write this as

$$x = \sum_{j=1}^{\infty} \big(u_j + \sum_n x_{j,n}\big) + \sum_{i \in I_0} x_i$$

where $x_{j,n} \in \ell_{p_{\langle j,n\rangle}}$ and I_0 is the set of indices which does not belong to any I_j's, $j = 1, 2, \ldots$.

Using the Mazur maps we define $\psi_{j,n} : \big(\ell_{p_{\langle j,n\rangle}} \oplus \ell_{p_{\langle j,n+1\rangle}}\big)_{c_0} \to (\ell_{q_j} \oplus \ell_{q_j})_{q_j}$ by

$$\psi_{j,n}(x_{j,n}, x_{j,n+1}) = \big(\varphi_{j,n}(x_{j,n}), \varphi_{j,n+1}(x_{j,n+1})\big).$$

Then $\psi_{j,n}$'s are equi-uniform homeomorphisms between 2^n-balls of the domain and $2^{1/q_j}2^n$-balls of the range. Let $\omega(\varepsilon)$ denote the common bound of moduli of continuity of $\varphi_{j,n}$ and $\psi_{j,n}$ and of their inverses.

For $n = 0, 1, \ldots$ put $\alpha_n = 2^n - 1$. For $\alpha_n \le t \le \alpha_{n+1}$ define

$$\tilde{g}_t(x) = \sum_{j=1}^{\infty} \Big[\psi_{j,n}^{-1}\big(h_{2^{-n}(t-\alpha_n)}(u_j, \psi_{j,n}(x_{j,n}, x_{j,n+1}))\big) + \sum_{i \ne n,n+1} x_{j,i}\Big] + \sum_{i \in I_0} x_i.$$

Here for each j, the map only replaces three coordinates in the block $(u_j, \ldots, x_{j,n}, x_{j,n+1}, \ldots)$ by $(\ldots, y_{j,n}, y_{j,n+1}, \ldots)$ where $(y_{j,n}, y_{j,n+1}) = \psi_{j,n}^{-1}\big(h_{2^{-n}(t-\alpha_n)}(u_j, \psi_{j,n}(x_{j,n},$ Note that for $t = \alpha_n$, $\tilde{g}_t$ is defined twice, however, the two definitions using $h_1^j(u_j, \psi_{j,n-1}(x_{j,n-1}, x_{j,n}))$ and the one using $h_0^j(u_j, \psi_{j,n}(x_{j,n}, x_{j,n+1}))$ coincide, therefore it is well-defined.

It is clear from the definition that $\tilde{g}_t$'s are homogeneous. It remains to check $\{\tilde{g}_t : 0 \le t < \infty\}$ is a family of equi-uniform homeomorphisms. This will imply the same for the norm-preserving g_t's (see the remarks before Theorem 10.28, [BL]).

Let $x = \sum_{j=1}^{\infty} \big(u_j + \sum_n x_{j,n}\big) + \sum_{i \in I_0} x_i$ and $y = \sum_{j=1}^{\infty} \big(v_j + \sum_n y_{j,n}\big) + \sum_{i \in I_0} y_i$ be in X_A such that $\|x\|, \|y\| \le \alpha_{n+1}$ and $\|x - y\| < 1$, and let $\alpha_n \le t, s \le \alpha_{n+1}$. Then $\|\tilde{g}_t(x) - \tilde{g}_s(y)\|$ is bounded by

$$\Big\| \sum_j \Big[\psi_{j,n}^{-1}\Big(h_{2^{-n}(t-\alpha_n)}(u_j, \psi_{j,n}(x_{j,n}, x_{j,n+1}))\Big) - \psi_{j,n}^{-1}\Big(h_{2^{-n}(s-\alpha_n)}(v_j, \psi_{j,n}(y_{j,n}, y_{j,n+1}))\Big)\Big] \Big\|_{c_0}$$

$$+ \Big\| \sum_j \sum_{i \ne n,n+1} (x_{j,i} - y_{j,i}) + \sum_{i \in I_0} (x_i - y_i) \Big\|_{c_0}.$$

The second term is bounded by $\|x - y\|$. By Lemma 7.5, for all j,

$$\Big\|\psi_{j,n}^{-1}\Big(h_{2^{-n}(t-\alpha_n)}(u_j, \psi_{j,n}(x_{j,n}, x_{j,n+1}))\Big) - \psi_{j,n}^{-1}\Big(h_{2^{-n}(s-\alpha_n)}(v_j, \psi_{j,n}(y_{j,n}, y_{j,n+1}))\Big)\Big\|$$

is bounded by

$$\omega\big(K\big\{\|u_j - v_j\| + \omega(\|x_{j,n} - y_{j,n}\| + \|x_{j,n+1} - y_{j,n+1}\|)\big\} + \alpha_{n+1}|2^{-n}s - 2^{-n}t|\big),$$

where the constant K and the function ω is independent of j. Since $\varepsilon \le C\omega(\varepsilon)$ for some constant C and all $\varepsilon \le 1$, and since the estimates are independent of j's, it follows that there exist constants L_1 and L_2 such that

$$\|\tilde{g}_t(x) - \tilde{g}_s(y)\| \le L_1\omega\big(L_2\omega(\|x - y\|) + |s - t|\big).$$

The same estimates hold for the inverses as well. □

References

[AD] S. A. Argyros and P. Dodos, *Genericity and amalgamation of classes of Banach spaces*, Advances in Mathematics 209 (2007), no. 2, 666–748.

[AL] I. Aharoni and J. Lindenstrauss, *An extension of a result of Ribe*, Israel Journal of Mathematics 52 (1985), 59-64.

[B] G. Beer, *A Polish topology for the closed subsets of a Polish space*, Proceedings of the American Mathematical Society 113 (1991), no. 4, 1123–1133.

[Ben] Y. Benyamini, *The uniform classification of Banach spaces*, Texas functional analysis seminar 1984–1985 (Austin, Tex.), 15–38, Longhorn Notes, Univ. Texas Press, Austin, TX, 1985. Also at arXiv math.FA/9406215.

[BK] H. Becker and A. S. Kechris, The Descriptive Set Theory of Polish Group Actions. London Mathematical Society Lecture Notes Series 232. Cambridge University Press, Cambridge, 1996.

[BL] Y. Benyamini and J. Lindenstrauss, Geometric Nonlinear Functional Analysis, Volume 1. American Mathematical Society Colloquium Publications 48, American Mathematical Society, Providence, RI, 2000.

[Bou] J. Bourgain, *Remarks on the extension of Lipschitz maps defined on discrete sets and uniform homeomorphism*, in Israel GAFA Seminar 1985-86 (J. Lindenstrauss and V. D. Milman, eds.), 157–167. Lecture Notes in Mathematics 1267, Springer, 1987.

[Bos] B. Bossard, *A coding of separable Banach spaces. Analytic and coanalytic families of Banach spaces*, Fundamenta Mathematicae 172 (2002), no. 2, 117–152.

[DH] R. Dougherty and G. Hjorth, *Reducibility and nonreducibility between ℓ_p equivalence relations*, Transactions of the American Mathematical Society 351 (1999), no. 5, 1835-1844.

[DJK] R. Dougherty, S. Jackson, and A. S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Transactions of the American Mathematical Society 341 (1994), no. 1, 193-225.

[FG] V. Ferenczi and E. M. Galego, *Some equivalence relations which are Borel reducible to isomorphism between separable Banach spaces*, Israel Journal of Mathematics 152 (2006), 61–82.

[FLR] V. Ferenczi, A. Louveau, and C. Rosendal, *The complexity of classifying separable Banach spaces up to isomorphism*, Journal of the London Mathematical Society, to appear.

[FR1] V. Ferenczi and C. Rosendal, *On the number of non-isomorphic subspaces of a Banach space*, Studia Mathematica 168 (2005), no. 3, 203–216.

[FR2] V. Ferenczi and C. Rosendal, *Complexity and homogeneity in Banach spaces*, in Banach Spaces and Their Applications in Analysis, 83–110. Walter de Gruyter, Berlin, 2007.

[Ga1] S. Gao, *Equivalence relations and classical Banach spaces*, in Mathematical Logic in Asia (Novosibirsk, Russia, 2005), 70-89. World Scientific, Hackensack, NJ, 2006.

[Ga2] S. Gao, Invariant Descriptive Set Theory. Monographs and Textbooks in Pure and Applied Mathematics 293, CRC Press, 2008.

[GK] S. Gao and A. S. Kechris, *On the classification of Polish metric spaces up to isometry*, Memoirs of the American Mathematical Society 161 (2003), no. 766.

[Go] E. Gorelik, *The uniform nonequivalence of L_p and ℓ_p*, Israel Journal of Mathematics 87 (1994), 1–8.

[HaKL] L. A. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, Journal of the American Mathematical Society 3 (1990), no. 4, 903–928.

[He] C. Hess, *Loi de probabilité des ensembles aléatoires à valeurs fermées dans un espace metrique séparable*, Comptes Rendus de l'Académie des Sciences Série I Mathématique 296 (1983), 883-886.

[HK] G. Hjorth and A. S. Kechris, *Borel equivalence relations and classifications of countable models*, Annals of Pure and Applied Logic 82 (1996), no. 3, 221-272.

[HjKL] G. Hjorth, A. S. Kechris, and A. Louveau, *Borel equivalence relations induced by actions of the symmetric group*, Annals of Pure and Applied Logic 92 (1998), 63-112.

[JLS] W. B. Johnson, J. Lindenstrauss, and G. Schechtman, *Banach spaces determined by their uniform structure*, Geometric Functional Analysis 6 (1996), 430–470.

[K] A. S. Kechris, Classical Descriptive Set Theory. Graduate Texts in Mathematics 156, Springer-Verlag, New York, 1995.

[KL] A. S. Kechris and A. Louveau, *The classification of hypersmooth Borel equivalence relations*, Journal of the American Mathematical Society 10 (1997), no. 1, 215-242.

[LR] A. Louveau and C. Rosendal, *Complete analytic equivalence relations*, Transactions of the American Mathematical Society 357 (2005), no. 12, 4839–4866.

[Me] J. Melleray, *Computing the complexity of the relation of isometry between separable Banach spaces*, Mathematical Logic Quarterly 53 (2007), no. 2, 128–131.

[Mi] B. S. Mityagin, *The homotopy structure of the linear group of a Banach space*, Russian Mathematical Surveys 25 (1970), no. 5, 59–103.

[P] A. Pełczyński, *Universal bases*, Studia Mathematica 32 (1969), 247–268.

[Ri] M. Ribe, *On uniformly homeomorphic normed spaces*, Arkiv för Matematik 14 (1976), no. 2, 237–244.

[Ri2] M. Ribe, *Existence of separable uniformly homeomorphic nonisomorphic Banach spaces*, Israel Journal of Mathematics 48 (1984), 139-147.

[Ro] C. Rosendal, *Cofinal families of Borel equivalence relations and quasiorders*, Journal of Symbolic Logic 70 (2005), no. 4, 1325–1340.

[So] S. Solecki, *Analytic ideals*, Bulletin of Symbolic Logic 2 (1996), no. 3, 339-348.

[T] N. Tomczak-Jaegermann, Banach-Mazur Distance and Finite-Dimensional Operator Ideals. Pitman Monographs and Surveys in Pure and Applied Mathematics, 38, 1989.

[W] N. Weaver, Lipschitz Algebras. World Scientific Publishing, River Edge, NJ, 1999.

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