Uncle Bill the Billionaire

The purpose of this project is to demonstrate how concepts from linear algebra can be applied. Follow the steps below.

An eccentric billionaire Bill tells his niece Nancy that he wishes to start distributing some of his wealth to his relatives before he dies in order to avoid inheritance taxes. Bill gives Nancy a choice as to how she may receive her money. The first choice is for Nancy to receive no money the first week, one cent the second week, and each week after the first two she is to receive five times the amount she received the week before minus 6 times what she received two weeks before. Her second choice is to receive no money the first week, one dollar the second week, and each week thereafter to receive the sum of the amounts she received the previous two weeks. Bill plans to make the same offer to all ten of his nephews and nieces. He will continue paying each one according to the plan each picks until he exhausts a supply of a billion dollars he set aside.

Nancy knows you are currently taking linear algebra. She therefore naturally asks you for your advice. Of course, Nancy wishes to get as much of the billion as she can. (She never did like her cousins and she has no sibling.) Follow the steps below to try to determine which choice she should take.

1. Let x_n and y_n be the amounts Nancy is to be given in the n^{th} week assuming the two choices. Write the formulas which determine the sequences $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$
2. Consider first Nancy's first option. Instead of paying no money the first week and one cent the second week, Bill could have started by paying any amount the first week and any other amount the second week and then continued according to his plan. For each choice of amounts paid the first two weeks one would obtain a sequence. Show that the set of all such sequences forms a vector space.
3. Find the dimension of the vector space in Part 2 and prove you are right.
4. Find all vectors in part 2 that are of the form $x_n = r^n$. Can you find a basis of vectors of this form. If so, prove it. If not, find a basis using as many vectors of the form $x_n = r^n$ as you can.
5. Based on your results, determine a formula for how much money Nancy will receive each week if she chooses the first option.
6. Do the same for the second option. You need not do the corresponding proofs in this case, just do the calculations.
7. Which choice will be better for Nancy?