

The Exponential Function

The Story

Speed O’Fender, a TAMS student, just received his driver’s license. He talked his father into letting him drive a car back to TAMS after a closed weekend. He noticed that just as he passed the 60 mile marker, the speedometer read exactly 60 miles per hour. A mile later he passed the 61 mile marker and he happened to be traveling at 61 miles per hour. Speed decided that it would be fun to always travel at the speed corresponding to the mile marker. For example, when he was at the 61.5 mile point, his speed was 61.5 miles per hour. When he reached the 72 mile marker, he was traveling exactly 72 miles per hour. When he was at the 80 mile marker, he was introduced to a member of the Texas State Police who did not have enough mathematical or scientific curiosity to encourage Speed in this experiment.

This experience suggested several interesting problems to Speed. First, is the function $s(t)$ that describes Speed’s position at time t an exponential function? Second, could this idea be used to define the exponential function instead of defining it as the inverse of the logarithm function? Third, if we define the exponential function using this idea, could we derive the usual properties of the exponential function? Fourth, is a minimization problem: How should Speed tell his parents that he got a ticket for speeding in order to minimize the negative consequences?

Speed went to you for help with these problems. Your job is to follow the outline below to help Speed understand how to define the exponential function using a simple differential equation.

Background Information

It will be helpful to use a theorem from differential equations that we will not cover in class. The theorem is a special case of an existence and uniqueness theorem that you will learn when you take a differential equations class.

Theorem 1

For any real numbers a and b , there is exactly one solution to the initial value problem

$$\begin{aligned}y' &= y, \\ y(a) &= b.\end{aligned}$$

Furthermore, the domain of the solution is all real numbers.

Example The function $f(x)$ satisfies

$$\begin{aligned}f'(x) &= f(x) \quad \text{for all } x, \\ f(10) &= 0.\end{aligned}$$

Show that $f(x) = 0$ for every real number x .

Solution Since $f'(x) = f(x)$, $f(x)$ satisfies the differential equation $y' = y$. So $f(x)$ satisfies the initial value problem:

$$\begin{aligned}y' &= y \\ y(10) &= 0.\end{aligned}$$

Let $g(x) = 0$ for every x . Then $g'(x) = 0 = g(x)$ for every x . We see that $g(x)$ satisfies the same initial value problem:

$$\begin{aligned}y' &= y \\ y(10) &= 0.\end{aligned}$$

The uniqueness part of Theorem 1 says that there is only one solution to this initial value problem. Therefore, $f(x) = g(x) = 0$ for every real number x . \triangle

Some initial value problems have more than one solution, while others have no solution. An example is the differential equation $x^3y' = y$. With the initial condition $y(0) = 1$ there is no solution, but with the initial condition $y(0) = 0$ there are infinitely many solutions.

Directions

Follow the outline below to give a reasonable definition of e^x . From high school algebra, we know how to define e^x for any rational number. The problem is that we cannot define e^x for irrational numbers using only precalculus mathematics. In the steps below, you will define the function $E(x)$ and derive some of its basic properties. In the end you will conclude that it is reasonable to define e^x to be $E(x)$ for all numbers, rational or irrational. Be careful not to assume anything about the function $E(x)$ other than its definition and what you have already proved about it.

You do not have to follow the steps below in exactly the order given. The goal is simply to define $E(x)$ as stated below and then derive the formula in step 10. If you wish to do things in a different order, that is fine. However, keep in mind that the outline was designed to allow you to break the bigger problem in smaller parts. For most of the parts, you will probably wish to use what you proved in a previous part.

1. Explain why Speed's position is modeled by the differential equation $y' = y$. We normally use t as the variable representing time, but for the remainder of the project let's use x . So y is a function of x .
2. Let's define $E(x)$ to be the unique solution to

$$\begin{aligned}y' &= y, \\ y(0) &= 1.\end{aligned}$$

Explain why $E'(x) = E(x)$. What is $E(0)$? Find a formula for $E''(x)$.

3. Use the chain rule to determine the derivative $\frac{d}{dx}E(u)$, where u is a function of x .

4. Prove that

$$E(a+x) = E(a)E(x),$$

where a and x are any numbers. (Hint: Think of a as a fixed number and x as a variable. Use the uniqueness part of Theorem 1.)

5. Use induction to prove that $E(nx) = (E(x))^n$ for any positive integer n and any real number x . (Hint: Fix x and then do an induction proof.)

6. Now, let's define $e = E(1)$. Prove that $E(n) = e^n$ for all positive integers n .

7. Prove that $E(-x) = \frac{1}{E(x)}$ for any real number x . Conclude that $E(-nx) = \frac{1}{E(x)^n}$ for every positive integer n and every real number x . (Hint: This one is not hard!)

8. Prove that $E(x) > 0$ for all x . Where is $E(x)$ increasing? Where is $E(x)$ concave up? (Hint: Use the uniqueness part of Theorem 1 to show $E(x) \neq 0$ for any value of x . You may wish to read the example after Theorem 1 again.)

9. Prove that for any positive integer n and any number x , $E\left(\frac{x}{n}\right) = \sqrt[n]{E(x)}$. Recall that for $r > 0$ and n a positive integer, $\sqrt[n]{r}$ is defined to be the unique positive number whose n^{th} power is r .

10. Prove that $E\left(\frac{p}{q}\right) = e^{\frac{p}{q}}$ for any integers p and $q \neq 0$.

11. Explain why it is reasonable to define $e^x = E(x)$ for every real number x .