

Summation Formulas

Purpose. You often see formulas in books for the sum of the first n positive integers, the sum of the first n squares, or the sum of the first powers of k for some integer k . Here you will learn how to derive these formulas.

The story. Shaggy does not know calculus, but he wants to know what the area under the curves $y = x^2$, $y = x^3$, and $y = x^k$ in general are. He set up sums that he thought should give good approximations to the areas (Riemann sums) and noticed that in order to estimate these areas using sums he had to compute sums that look like

$$\sum_{k=0}^n k^r.$$

Scooby was able to provide the correct formulas for Shaggy, but Scooby did not remember how to derive the formulas. Scooby said the derivation did not matter, since one could prove the formulas by induction. But Shaggy, being the scholar that he is, was not satisfied with this since as he put it “Like, how did they ever find these formulas in the first place?”

Your job is to help Shaggy and Scooby solve the mystery of how to derive these formulas.

Procedure. You are to follow the outline below.

1. Define $F_k(x) = x(x-1)(x-2)\cdots(x-k+1)$ for $k = 1, 2, 3, \dots$ and $F_0(x) = 1$. Write out the formulas for $F_k(x)$ for $k = 0, 1, 2, 3, 4$. Expand $F_k(0)$ for each $k = 0, 1, 2, 3, \dots$
2. Prove that every polynomial $g(x)$ of degree n can be written in the form $g(x) = a_0F_0(x) + a_1F_1(x) + a_2F_2(x) + \cdots + a_nF_n(x)$ for some numbers a_0, a_1, \dots, a_n . (Hint: Try induction.)
3. Given a function $g(x)$, define $\Delta g(x) = g(x+1) - g(x)$. This is called the *forward difference* for g .
 - a. Find a simple formula for $\Delta F_k(x)$. (Your answer should simplify a lot!)
 - b. We define $(af + bg)(x)$ to be $af(x) + bg(x)$. Show that $\Delta(ag + bf)(x) = a\Delta g(x) + b\Delta f(x)$ for any functions f and g and any constants a and b .
4. Prove that if $g(x)$ is a polynomial of degree n , then $\Delta g(x)$ is a polynomial of degree $n - 1$.

For the rest of this project consider the functions to have domain $\{0, 1, 2, 3, \dots\}$ only. Note that this assumption allows you to prove facts about $f(x)$ using induction on x .

5. Prove that if $\Delta f(x) = \Delta g(x)$ for all $x \in \{0, 1, 2, 3, \dots\}$, then $f(x) = g(x) + C$ for some constant C .
6. Prove that if $g(x)$ is a function such that $\Delta g(x)$ is a polynomial of degree k , then $g(x)$ is a polynomial of degree $k + 1$. (Hint: Use induction on the degree of $\Delta g(x)$.)
7. Let

$$f_r(n) = \sum_{k=0}^n k^r.$$

Use part 6) to show that $f_r(n)$ is a polynomial in n . What is the degree of f_r ? (If you are having trouble understanding what $f_r(n)$ means, write out a few examples, like $f_3(5)$.)

8. Let $\Delta^t g(x)$ be the t^{th} forward difference for $g(x)$. In other words, it is just the result from applying the forward difference t times. Use part 3) and the forward difference of g to find formulas for the coefficients a_k . (Hint: Try substituting $x = 0$. Your answer should involve things like $\Delta^t g(0)$.)
9. Use what you have learned above to derive the formulas for $f_r(n)$ for $r = 1, 2, 3, 4, 5, 6$.
10. Use what you learned to derive the leading coefficient of $f_r(n)$ for any value of r . Then use this fact to show $\int_0^1 x^r dx = \frac{1}{r+1}$ for $r = 1, 2, 3, \dots$ without using the Fundamental Theorem of Calculus.