A–1 Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

A–2 What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor ?$$

A–3 Evaluate $\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2+n+1)$, where $\operatorname{Arccot} t$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot\theta = t$.

A–4 A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

(a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.

(b) The sum of the n entries of a transversal is the same for all transversals of A.

An example of such a matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,$$

where the a_i's and b_i's are rational numbers.

A–5 Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are functions of n real variables $x = (x_1, \ldots, x_n)$ with continuous second-order partial derivatives everywhere on $\mathbf{R}^n$. Suppose further that there are constants c_{ij} such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$

for all i and j, $1 \leq i \leq n$, $1 \leq j \leq n$. Prove that there is a function $g(x)$ on $\mathbf{R}^n$ such that $f_i + \partial g/\partial x_i$ is linear for all i, $1 \leq i \leq n$. (A linear function is one of the form

$$a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.)$$

A–6 Let $a_1, a_2, \ldots, a_n$ be real numbers, and let $b_1, b_2, \ldots, b_n$ be distinct positive integers. Suppose that there is a polynomial $f(x)$ satisfying the identity

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1, b_2, \ldots, b_n$ and n (but independent of $a_1, a_2, \ldots, a_n$).