A–1 Curves $A, B, C$ and $D$ are defined in the plane as follows:

$$A = \left\{ (x, y) : x^2 - y^2 = \frac{x}{x^2 + y^2} \right\},$$

$$B = \left\{ (x, y) : 2xy + \frac{y}{x^2 + y^2} = 3 \right\},$$

$$C = \left\{ (x, y) : x^3 - 3xy^2 + 3y = 1 \right\},$$

$$D = \left\{ (x, y) : 3x^2y - 3x - y^3 = 0 \right\}.$$

Prove that $A \cap B = C \cap D$.

A–2 The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the $10^n$-th digit in this sequence occurs in the part of the sequence in which the $m$-digit numbers are placed, define $f(n)$ to be $m$. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

A–3 For all real $x$, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$  

(a) If $f(x) > 0$ for all real $x$, must $f'(x) > 0$ for all real $x$? Explain.
(b) If $f'(x) > 0$ for all real $x$, must $f(x) > 0$ for all real $x$? Explain.

A–4 Let $P$ be a polynomial, with real coefficients, in three variables and $F$ be a function of two variables such that

$$P(ux, uy, uz) = u^2F(y - x, z - x)$$

for all real $x, y, z, u$, and such that $P(1, 0, 0) = 4$, $P(0, 1, 0) = 5$, and $P(0, 0, 1) = 6$. Also let $A, B, C$ be complex numbers with $P(A, B, C) = 0$ and $|B - A| = 10$. Find $|C - A|$. 
A–5 Let
\[ \vec{G}(x, y) = \left( \frac{-y}{x^2 + 4y^2}, \frac{x}{x^2 + 4y^2}, 0 \right). \]

Prove or disprove that there is a vector-valued function
\[ \vec{F}(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z)) \]
with the following properties:
(i) \( M, N, P \) have continuous partial derivatives for all \((x, y, z) \neq (0, 0, 0)\);
(ii) \( \text{Curl} \vec{F} = \vec{0} \) for all \((x, y, z) \neq (0, 0, 0)\);
(iii) \( \vec{F}(x, y, 0) = \vec{G}(x, y) \).

A–6 For each positive integer \( n \), let \( a(n) \) be the number of zeroes in the base 3 representation of \( n \). For which positive real numbers \( x \) does the series
\[ \sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3} \]
converge?