A–1 Curves $A, B, C$ and $D$ are defined in the plane as follows:

$$\begin{aligned} A &= \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\right\}, \\ B &= \left\{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\right\}, \\ C &= \left\{(x,y) : x^3 - 3xy^2 + 3y = 1\right\}, \\ D &= \left\{(x,y) : 3x^2y - 3x - y^3 = 0\right\}. \end{aligned}$$

Prove that $A \cap B = C \cap D$.

A–2 The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the $10^n$-th digit in this sequence occurs in the part of the sequence in which the $m$-digit numbers are placed, define $f(n)$ to be $m$. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

A–3 For all real $x$, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real $x$, must $f'(x) > 0$ for all real $x$? Explain.

(b) If $f'(x) > 0$ for all real $x$, must $f(x) > 0$ for all real $x$? Explain.

A–4 Let $P$ be a polynomial, with real coefficients, in three variables and $F$ be a function of two variables such that

$$P(ux, uy, uz) = u^2 F(y - x, z - x) \quad \text{for all real } x, y, z, u,$$

and such that $P(1,0,0) = 4$, $P(0,1,0) = 5$, and $P(0,0,1) = 6$. Also let $A, B, C$ be complex numbers with $P(A,B,C) = 0$ and $|B - A| = 10$. Find $|C - A|$.

A–5 Let

$$\vec{G}(x,y) = \left( \frac{-y}{x^2+4y^2}, \frac{x}{x^2+4y^2}, 0 \right).$$

Prove or disprove that there is a vector-valued function

$$\vec{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$$

with the following properties:

(i) $M, N, P$ have continuous partial derivatives for all $(x,y,z) \neq (0,0,0)$;

(ii) $\operatorname{Curl} \vec{F} = \vec{0}$ for all $(x,y,z) \neq (0,0,0)$;

(iii) $\vec{F}(x,y,0) = \vec{G}(x,y)$.

A–6 For each positive integer $n$, let $a(n)$ be the number of zeroes in the base 3 representation of $n$. For which positive real numbers $x$ does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?