A–1 Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

A–2 Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

A–3 Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$\begin{aligned} a_m(0) &= d/2^m, \\ a_m(j+1) &= (a_m(j))^2 + 2a_m(j), \qquad j \geq 0. \end{aligned}$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

A–4 Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

A–5 Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)\,dx$. For which integers m, $1 \leq m \leq 10$ is $I_m \neq 0$?

A–6 If $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ is a polynomial with real coefficients a_i, then set

$$\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.$$

Let $F(x) = 3x^2 + 7x + 2$. Find, with proof, a polynomial $g(x)$ with real coefficients such that

(i) $g(0) = 1$, and

(ii) $\Gamma(f(x)^n) = \Gamma(g(x)^n)$

for every integer $n \geq 1$.