A–1 Determine, with proof, the number of ordered triples \((A_1, A_2, A_3)\) of sets which have the property that

(i) \(A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\), and
(ii) \(A_1 \cap A_2 \cap A_3 = \emptyset\).

Express your answer in the form \(2^a 3^b 5^c 7^d\), where \(a, b, c, d\) are nonnegative integers.

A–2 Let \(T\) be an acute triangle. Inscribe a rectangle \(R\) in \(T\) with one side along a side of \(T\). Then inscribe a rectangle \(S\) in the triangle formed by the side of \(R\) opposite the side on the boundary of \(T\), and the other two sides of \(T\), with one side along the side of \(R\).

For any polygon \(X\), let \(A(X)\) denote the area of \(X\). Find the maximum value, or show that no maximum exists, of \(\frac{A(R) + A(S)}{A(T)}\), where \(T\) ranges over all triangles and \(R, S\) over all rectangles as above.

A–3 Let \(d\) be a real number. For each integer \(m \geq 0\), define a sequence \(\{a_m(j)\}\), \(j = 0, 1, 2, \ldots\) by the condition

\[
a_m(0) = \frac{d}{2^m}, \quad a_m(j + 1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.
\]

Evaluate \(\lim_{n \to \infty} a_n(n)\).

A–4 Define a sequence \(\{a_i\}\) by \(a_1 = 3\) and \(a_{i+1} = 3^{a_i}\) for \(i \geq 1\). Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many \(a_i\)?

A–5 Let \(I_m = \int_0^{2\pi} \cos(x) \cos(2x) \cdots \cos(mx) \, dx\). For which integers \(m, 1 \leq m \leq 10\) is \(I_m \neq 0\)?

A–6 If \(p(x) = a_0 + a_1 x + \cdots + a_m x^m\) is a polynomial with real coefficients \(a_i\), then set

\[
\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.
\]

Let \(F(x) = 3x^2 + 7x + 2\). Find, with proof, a polynomial \(g(x)\) with real coefficients such that

(i) \(g(0) = 1\), and
(ii) \(\Gamma(f(x)^n) = \Gamma(g(x)^n)\)

for every integer \(n \geq 1\).