MATH 3000 Exam 1 February 14, 2014 Name____________________________________

You are not to use a calculator, book, notes, or others to do this exam.

1. Define the following. *(35 points)*

a. $N^*(x;\epsilon)$ where $x \in \mathbf{R}$ and $\epsilon > 0$

See your book & notes for definitions.

b. x is an accumulation point of the set $S \subseteq \mathbf{R}$

c. x is an interior point of the set $S \subseteq \mathbf{R}$

d. $S \subseteq \mathbf{R}$ is an open set

e. x is a boundary point of the set $S \subseteq \mathbf{R}$

f. $\bigcup_{\alpha \in I} A_\alpha$ where for each $\alpha \in I$, A_α is a set.

g. The closure of $S \subseteq \mathbf{R}$, $\mathrm{cl}(S)$

2. **Give the truth table for $(p \vee q) \Rightarrow (p \wedge q)$.** ***(10 points)***

p	q	$(p \vee q) \Rightarrow (p \wedge q)$
T	T	T
T	F	F
F	T	F
F	F	T

3. **Negate the following statements. For each part indicate if the original statement is true and if the negated statement is true.** ***(15 points)***

a. $\forall S \subseteq \mathbf{R}, \forall x \in \mathbf{R}, (x \in S') \Rightarrow (x \in \mathrm{bd}(S))$

$\exists S \subseteq \mathbb{R} \ni \exists x \in \mathbb{R} \ni (x \in S') \wedge (x \notin \mathrm{bd} S)$

Statement true?

Negation true?

b. $\forall x \in \mathbf{R}, \exists y \in \mathbf{R} \ni 0 \leq \frac{x}{y} \leq 1$

$\exists x \in \mathbb{R} \ni \forall y \in \mathbb{R}, (x/y < 0) \vee (x/y > 1)$

Statement true?

Negation true?

4. Let $S = \{\, 2 + \frac{1}{n} \mid n \in \mathbf{N}\} \cup [6, 10)$ *(10 points)*

a. Find $\text{bd}(S)$

$S \cup \{2, 6, 10\}$

b. Find S' $\{2\} \cup [6, 10]$

c. Find $\text{cl}(S)$ $S \cup \{2\} \cup [6, 10]$

d. Find $\text{int}(S)$ $(6, 10)$

5. Prove using the definitions, not lemmas or theorems proved in class, that for any set $S \subseteq \mathbf{R}$ and any $x \in \text{bd}(S) - S$, x is an accumulation point for S. *(10 points)*

Let $x \in \text{bd} S \setminus S$.

Let $\varepsilon > 0$.

Then $N(x; \varepsilon) \cap S \neq \emptyset$ since $x \in \text{bd}(S)$.

Since $x \notin S$, $x \notin N(x; \varepsilon) \cap S$.

$\therefore \; N^*(x; \varepsilon) \cap S = N(x; \varepsilon) \cap S$

$\therefore \; N^*(x; \varepsilon) \cap S \neq \emptyset$.

$\therefore \; x \in S'$.

6. **Prove that if $U, V \subseteq \mathbf{R}$ are both open, then $U \cap V$ is also open.** *(10 points)*

Let $x \in U \cap V$. We show $x \in \text{int}(U \cap V)$.

Since $x \in U$ and U is open, $x \in \text{int}(U)$. $\therefore \exists \varepsilon_1 > 0 \ni N(x;\varepsilon_1) \subseteq U$.

Since $x \in V$ and V is open, $x \in \text{int}(V)$ $\therefore \exists \varepsilon_2 > 0 \ni N(x;\varepsilon_2) \subseteq V$.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

$\therefore N(x;\varepsilon) \subseteq N(x;\varepsilon_1)$ & $N(x;\varepsilon) \subseteq N(x;\varepsilon_2)$

$\therefore N(x;\varepsilon) \subseteq U$ & $N(x;\varepsilon) \subseteq V$.

$\therefore N(x;\varepsilon) \subseteq U \cap V$ $\quad \therefore x \in \text{int}(U \cap V)$ $\therefore U \cap V$ is open.

7. **State whether each statement is true or false. If true, just state that it is true. If false, say it is false and then find a counterexample that shows it is false. You need not give explanation with your counterexamples. Keep in mind that I am asking if the statement is true, and not if it is the full definition of something.** *(15 points)*

a. Any intersection of open sets is open.

F $\qquad \displaystyle\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$

b. Any intersection of closed sets is closed.

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c. A set $C \subseteq \mathbf{R}$ is closed if and only if $\text{cl}(C) = C$.

T

d. If a set $S \subseteq \mathbf{R}$ is not open, then S is closed.

F $\qquad S = [0, 1)$

e. A set $S \subseteq \mathbf{R}$ is closed if and only if $\text{bd}(S) = S'$.

F $\qquad \text{bd}(\mathbb{Q}) = \mathbb{Q}' = \mathbb{R}$

MATH 3000 Exam 2 March 26, 2014 Name________________________

You are not to use a calculator, book, notes, or others to do this exam.

1. Define the following. *(20 points)*

   a. Let $S \subseteq \mathbf{R}$. Give the definition of an upper bound for S.

   b. Let $S \subseteq \mathbf{R}$. Give the definition of a maximum for S.

   See your notes or book for definitions and the completion axiom.

   c. $\sup(S)$ where $S \subseteq \mathbf{R}$.

   d. For $f : A \to B$, define what it means for f to be surjective.

   e. For $f : A \to B$ and $D \subseteq B$, define $f^{-1}(D)$.

2. State the completion axiom for the real numbers. *(5 points)*

3. Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^2$. Find the following sets. *(15 points)*

a. $f([2,3))$

$= [4,9)$

b. $f((-1,4]) = [0,4]$

c. $f^{-1}([16,25)) = (-5,-4] \cup [4,5)$

d. $f^{-1}((-\infty,-1)) = \emptyset$

4. Find the following sets. *(10 points)*

a. $\mathrm{cl}(\mathbf{Q}) = \mathbb{R}$

b. $\mathrm{int}(\mathbf{R} - \mathbf{Q}) = \emptyset$

5. **Prove that if $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f : A \to C$ is also injective.** *(10 points)*

Let $x_1, x_2 \in A$ and assume $g\circ f(x_1) = g\circ f(x_2)$.

Then $g(f(x_1)) = g(f(x_2))$

$\therefore$ $f(x_1) = f(x_2)$ since g is injective

$\therefore$ $x_1 = x_2$ since f is injective

$\therefore$ $g\circ f$ is injective.

6. Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = 2x^2 + 3x - 1$. Prove that f is continuous at 3. *(15 points)*

Let $\varepsilon > 0$ be arbitrary.

Let $\delta = \min\{1, \varepsilon/17\}$. $\delta > 0$ since $\varepsilon > 0$.

Let x satisfy $|x-3| < \delta$.

Then $|x-3| < 1$, so $2 < x < 4$

$\therefore$ $4 < 2x < 8$ & $13 < 2x+9 < 17$

$\therefore$ $|2x+9| < 17$.

Now, $|f(x) - f(3)| = |2x^2+3x-1-(18+9-1)|$

$$\begin{aligned} &= |2x^2+3x-27| \\ &= |(x-3)(2x+9)| \\ &= |x-3|\,|2x+9| \\ &\leq |x-3|\cdot 17 \\ &< \delta \cdot 17 \\ &\leq \tfrac{\varepsilon}{17}\cdot 17 \\ &= \varepsilon. \end{aligned}$$

$\therefore$ f is cont. at 3.

7. Let $f : A \to B$ and $C_1, C_2 \subseteq A$. Prove that $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$. (10 points)

Let $y \in f(C_1 \cup C_2)$. $\therefore$ $y = f(x)$ for some $x \in C_1 \cup C_2$.

$\therefore$ $x \in C_1$ or $x \in C_2$. $\therefore$ $y = f(x) \in f(C_1)$ or $y = f(x) \in f(C_2)$

$\therefore$ $y \in f(C_1) \cup f(C_2)$.

$\therefore$ $f(C_1 \cup C_2) \subseteq f(C_1) \cup f(C_2)$

Next let $y \in f(C_1) \cup f(C_2)$. Then $y \in f(C_1)$ or $y \in f(C_2)$.

$\therefore$ $y = f(x)$ for some $x \in C_1$ or $y = f(x)$ for some $x \in C_2$.

$\therefore$ $y = f(x)$ for some $x \in C_1 \cup C_2$.

$\therefore$ $y = f(x) \in f(C_1 \cup C_2)$.

$\therefore$ $f(C_1) \cup f(C_2) \subseteq f(C_1 \cup C_2)$.

8. State whether each statement is true or false. If true, just state that it is true. If false, say it is false and then find a counterexample that shows it is false. You need not give explanation with your counterexamples. Keep in mind that I am asking if the statement is true, and not if it is the full definition of something. (20 points)

a. There is open subset of the real numbers that contains 0, but it contains no irrational number.

F Every neighborhood of 0 contains an irrational number.

(Note that this statement has no counterexample due to the nature of the statement.)

b. If $f : A \to B$ and $D \subseteq B$, then $D = f(f^{-1}(D))$.

F $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ $D = (-\infty, 0)$ $f(f^{-1}(D)) = f(\emptyset) = \emptyset$.

c. For any $S \subseteq \mathbf{R}$, $S' \subseteq \text{bd}(S)$.

F $S = (0,1)$, $S' = [0,1]$, $\text{bd}(S) = \{0,1\}$.

d. If $S \subseteq \mathbf{R}$, $\sup(S) \in S$ and $\inf(S) \in S$, then S is closed.

F $S = [0,1) \cup (2,3]$.

e. A set $S \subseteq \mathbf{R}$ is open if and only if $S' - \text{bd}(S) = S$.

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MATH 3000 Exam 3 April 25, 2014 Name____________________________

You are not to use a calculator, book, notes, or others to do this exam.

1. Define the following. *(15 points)*

   a. Let $S \subseteq \mathbf{R}$. Give the definition of an open cover for S.

You can find the definitions and theorems in your notes or book.

   b. Let $\mathbf{U} = \{U_i \mid i \in I\}$ be a cover for S. Give the definition of a subcover of $\mathbf{U}$ for S.

   c. $S \subseteq \mathbf{R}$. Give the definition for S compact.

2. State the Heine-Borel Theorem. *(5 points)*

3. State the Bolzano-Weierstrass Theorem. *(5 points)*

4. State the Extreme Value Theorem. *(5 points)*

5. Determine which of the following sets is compact. Explain how you know. You can either use the definition to explain, or you can use a theorem such as the Heine-Borel or Bolzano-Weierstrass Theorem. *(15 points)*

a. $[2,3)$ Not closed $\therefore$ Not compact.

b. $\{\frac{1}{n} | n \in \mathbf{N}\}$ Not closed $\therefore$ Not compact

c. $[0,\infty)$ Not bounded $\therefore$ Not compact

d. $[-1000,1000] - \cup_{n=1}^{\infty}(-100,n)$ $= [-1000, 100]$ closed & bounded $\therefore$ compact.

6. The Nested Interval Theorem states that if $\forall n \in \mathbf{N}, I_n$ is a nonempty closed and bounded interval and $I_{n+1} \subseteq I_n$, then $\cap_{n\in\mathbf{N}} I_n \neq \emptyset$. Give examples to show that the conclusion of the theorem is not true if all the conditions of the theorem are satisifed except the given one. *(10 points)*

a. The sets I_n are all open instead of closed.

$I_n = (0, \frac{1}{n})$

$\bigcap_{n\in\mathbb{N}} I_n = \emptyset.$

b. The sets I_n are allowed to be unbounded.

$I_n = [n, \infty)$

$\bigcap_{n\in\mathbb{N}} I_n = \emptyset.$

7. Use induction to prove that for all natural numbers n, $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)} = \frac{n}{n+1}$. (10 points)

$\forall n \in \mathbb{N}$, let $S(n)$ be the statement that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} = \frac{n}{n+1}.$$

The statement $S(1)$ is $\frac{1}{1\cdot 2} = \frac{1}{1+1}$ which is true since both sides are $\frac{1}{2}$.

Now we assume $n \in \mathbb{N}$ and $S(n)$ is true. We show $S(n+1)$ is true.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}+\frac{1}{(n+1)((n+1)+1)} = \frac{n}{n+1}+\frac{1}{(n+1)(n+2)}$$ by the inductive assumption.

$$= \frac{1}{n+1}\left[n+\frac{1}{n+2}\right] = \frac{1}{n+1}\left[\frac{n^2+2n}{n+2}+\frac{1}{n+2}\right]$$

$$= \frac{1}{n+1}\,\frac{(n+1)^2}{n+2} = \frac{n+1}{n+2} = \frac{n+1}{(n+1)+1}$$

$\therefore S(n+1)$ is true.

$\therefore \forall n \in \mathbb{N}, S(n)$ is true.

8. Prove the set $(0, 1]$ is not compact using the definition of compact. (Little credit will be given if you use a major theorem like the Heine-Borel Theorem.) (10 points)

Let $\mathcal{U} = \{(\frac{1}{n}, \infty) \mid n \in \mathbb{N}\}$. Each set $(\frac{1}{n},\infty) \in \mathcal{U}$ is open and $\bigcup_{n\in\mathbb{N}} (\frac{1}{n}, \infty) = (0,\infty)$ which contains $(0,1]$.

$\therefore \mathcal{U}$ is an open cover for $(0,1]$.

We show that $\mathcal{U}$ has no finite subcover using proof by contradiction. If $\mathcal{U}' \subseteq \mathcal{U}$ is a finite subcover for S,

then $(0,1] \subseteq (\frac{1}{1},\infty) \cup (\frac{1}{2},\infty) \cup \cdots \cup (\frac{1}{n},\infty)$ for some $n \in \mathbb{N}$.

But $(\frac{1}{1},\infty) \cup \cdots \cup (\frac{1}{n},\infty) = (\frac{1}{n},\infty)$ and $\frac{1}{n} \notin (\frac{1}{n},\infty)$

Since $\frac{1}{n} \in (0,1]$, $(0,1] \nsubseteq (\frac{1}{n},\infty)$ $\#$

$\therefore \mathcal{U}$ has no finite subcover.

$\therefore \mathcal{U}$ is not compact.

9. Use the definition of compact to prove that $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is compact. (Little credit will be given if you use a major theorem like the Heine-Borel Theorem.) *(15 points)*

Let $\mathcal{U}$ be any open cover for $S = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$.
Since $\mathcal{U}$ covers S, $\exists$ sets $U_0, U_1, U_2, U_3, \dots$ in $\mathcal{U}$ with $0 \in U_0$, $\frac{1}{1} \in U_1$, $\frac{1}{2} \in U_2$, $\frac{1}{3} \in U_3, \dots$

Since U_0 is open and $0 \in U_0$, 0 is an interior point of U_0. $\therefore \exists \varepsilon > 0 \ni N(0;\varepsilon) \subseteq U_0$.

$\exists N \in \mathbb{N} \ni \frac{1}{N} < \varepsilon$.

Next we show $S = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq U_0 \cup U_1 \cup \dots \cup U_{N-1}$.
Let $x \in S$. Then either $x = 0$ or else $x = \frac{1}{n}$ for some $n \in \mathbb{N}$.
If $x = 0$, $x \in U_0$. So suppose $x = 1/n$.
We look at ~~two~~ cases:
Case 1: $n \geq N$. Then $\frac{1}{n} < \frac{1}{N} < \varepsilon$ $\therefore x = \frac{1}{n} \in N(0;\varepsilon)$
$\therefore x \in U_0$.
Case 2: $n < N$. Then $x = \frac{1}{n} = U_n$ and $n \leq N-1$.

In in any case, $x \in U_0 \cup U_1 \cup U_2 \cup \dots \cup U_{N-1}$.
$\therefore \mathcal{U}$ has a finite subcover for S, namely
$\{U_0, U_1, U_2, \dots, U_{N-1}\}$.

10. State whether each statement is true or false. If true, just state that it is true. If false, say it is false and then find a counterexample that shows it is false. You need not give explanation with your counterexamples. Keep in mind that I am asking if the statement is true, and not if it is the full definition of something. *(15 points)*

a. There is an infinite set S with $S \subseteq (-1000, 1000)$ and $S' = \emptyset$.

F by Bolzano-Weierstrass.

b. The intersection of any collection of compact subsets of $\mathbf{R}$ is compact.

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c. If $f : A \to \mathbf{R}$ is continuous and $A \subseteq \mathbf{R}$ is closed, then $f(A)$ is closed.

F

$f : (0, \infty) \to \mathbb{R}$

$f(x) = \frac{1}{x}$

$A = [1, \infty)$

d. If $f : A \to \mathbf{R}$ is continuous and $A \subseteq \mathbf{R}$ is bounded, then $f(A)$ is bounded.

F

$f : (0, 1] \to \mathbb{R}$

$f(x) = \frac{1}{x}$

$f((0,1]) = [1, \infty)$.

e. If $f : A \to \mathbf{R}$ is continuous and $A \subseteq \mathbf{R}$ is closed and bounded, then $f(A)$ is closed and bounded.

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