

This set of exercises will show that if the algorithm we developed to find the left inverse of a square matrix succeeds, then the matrix we obtain is in fact the inverse. Recall that we augment a square matrix \mathbf{A} with the identity matrix and put the augmented matrix in reduced row echelon form. If we get the identity matrix augmented by \mathbf{B} , then we saw that $\mathbf{AB} = \mathbf{I}$. Now we wish to show that $\mathbf{A}^{-1} = \mathbf{B}$. In other words, \mathbf{B} is not just a “right” inverse but also a “left” inverse for \mathbf{A} .

Complete the following exercises to show that if \mathbf{A} is an $n \times n$ matrix, then the algorithm we discussed in class actually gives the inverse for \mathbf{A} , and not just a right inverse. Assume that each matrix in these exercises is square.

1. Show that if $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_k$ all have inverses, then $\mathbf{C} = \mathbf{E}_1\mathbf{E}_2 \dots \mathbf{E}_k$ has an inverse by guessing the formula for \mathbf{C}^{-1} involving $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_k$ and then checking the definition of inverse.
2. Let $1 \leq i \leq n$ and $1 \leq j \leq n$ with $i \neq j$ and let \mathbf{E} be the $n \times n$ matrix one obtains by switching the i^{th} and j^{th} rows of the $n \times n$ identity matrix. Do some examples to determine the effect of multiplying \mathbf{EA} for an $n \times n$ matrix \mathbf{A} . Explain the connection with row operations.
3. Let \mathbf{E} be as in 2). Find \mathbf{E}^2 . Does \mathbf{E} have an inverse? If so, what is its inverse?
4. Let $r \neq 0$ be a real number and $1 \leq i \leq n$ be an integer. Define \mathbf{F} to be the matrix obtained by multiplying the i^{th} row of the $n \times n$ identity matrix by r . Do some examples to determine the effect of multiplying \mathbf{FA} where \mathbf{A} is an $n \times n$ matrix. Explain the connection with row operations.
5. Let \mathbf{F} be as in 4). Guess the inverse of \mathbf{F} and verify that the matrix you found is the inverse.
6. Let $1 \leq i \leq n$ and $1 \leq j \leq n$ with $i \neq j$. Let r be a real number and let \mathbf{G} be the $n \times n$ matrix obtained from the identity matrix by adding r times row i to row j . Do some examples to determine the effect of multiplying \mathbf{GA} , where \mathbf{A} is an $n \times n$ matrix. Explain the connection with row operations. Guess what \mathbf{G}^{-1} is and verify that your guess is correct.
7. Suppose that \mathbf{A} is an $n \times n$ matrix and it can be row reduced to the identity matrix. Explain why $\mathbf{C}_1\mathbf{C}_2 \dots \mathbf{C}_k\mathbf{A} = \mathbf{I}$ for some collection of invertible matrices $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k$, each of the form of a matrix defined above.
8. Use matrix algebra on the formula in part 7) to find a formula for \mathbf{A}^{-1} . Explain why the algorithm we developed to find an inverse actually gives the inverse when it succeeds.