

1. Solve the matrix equation

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -8 \end{bmatrix}.$$

If the solution is not unique, then write your answer in parametric form.

2. Is the vector  $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$  an element of

$\text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 8 \\ 8 \end{bmatrix}\right)$ ? If so,

find the coefficients (or weights). If not state how you know.

3. Find the parametric form of the equation of the line through

$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$  and parallel with the vector

$\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ .

4. In which of the situations can you say if the vectors that make up the columns of the matrix  $\mathbf{A}$  are linearly independent or linearly dependent? Explain your answer.

- $\mathbf{A}$  is  $3 \times 6$ .
- $\mathbf{A}$  is  $6 \times 3$ .
- Each row of  $\mathbf{A}$  sums to zero.
- Each column of  $\mathbf{A}$  sums to zero.
- When row reduced, each row has a pivot element.

- f) When row reduced, each column has a pivot element.

5. Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}$$

- Does the set  $S$  span  $\mathbf{R}^3$ ? Justify your answer.
- Is  $S$  a linearly independent set? Justify your answer.

6. Give the definitions:

- A linear transformation  $T : V \rightarrow W$  where  $V$  and  $W$  are vector spaces.
- A spanning set for a vector space  $V$ .
- A linearly independent set of vectors in a vector space.
- The matrix for a linear transformation  $T : V \rightarrow W$  with respect to the basis  $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  for  $V$  and  $D = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$  for  $W$ .

7. Let  $T$  be the linear transformation of the plane that reflects a vector across the line  $y = x$  and then rotates the result by  $\pi$ . Find the standard matrix for  $T$  and use it to compute what  $T$  does to the vector  $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ .

8. Let  $\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$  and  $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be given by  $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$ .

- a) Determine if  $T$  is one-to-one. Explain.  
 b) Determine if  $T$  is onto. Explain.

9. Compute the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

or else explain why the inverse does not exist.

10. Suppose that  $\mathbf{AB} = \mathbf{I}$ , where each matrix is  $n \times n$ . Show that for any vector  $b \in \mathbf{R}^n$ , the equation  $\mathbf{A}x = b$  has a solution.
11. Suppose all the entries in the second row of  $\mathbf{B}$  are 0. What (if anything) does this say about the entries of  $\mathbf{AB}$ ? What about  $\mathbf{BA}$ ?
12. Suppose that  $\mathbf{C}$  and  $\mathbf{B}$  are invertible and that  $\mathbf{A} = \mathbf{CBC}^{-1}$ . Find  $\mathbf{A}^{-1}$  and verify that your answer is correct.
13. A linear transformation  $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  is given by the formula

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y + 2z \\ x + y \\ y + z \end{bmatrix}.$$

Does the inverse  $T^{-1}$  exist? If so, compute the inverse. If not, say how you know.

14. Give the definition of a subspace of a vector space  $V$ .

15. a) Can a square matrix with two identical rows be invertible? Explain.  
 b) Can a square matrix with two identical columns be invertible? Explain.

16. Determine if the vectors form a basis for  $\mathbf{R}^3$ .

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 0 \end{bmatrix}$$

17. Find the dimension of the span of the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}.$$

18. Find a basis for the null space and the column space of the matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 3 & 8 \end{bmatrix}.$$

Give the dimensions of the null space and the column space.

19. If the rank of a  $7 \times 6$  matrix  $\mathbf{A}$  is 4, what is the dimension of the subspace of all solutions to  $\mathbf{A}x = 0$ ?

20. Compute  $\det \begin{bmatrix} 1 & -4 & 2 \\ -2 & 3 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ .

21. Compute  $\det \begin{bmatrix} -1 & 2 & 6 & 2 & 9 \\ 0 & 4 & 3 & 2 & 0 \\ 0 & 0 & 3 & -4 & 6 \\ 0 & 0 & 0 & -1 & 6 \\ 1 & -2 & -3 & 2 & -8 \end{bmatrix}$

22. The determinant of a matrix does not change when the first row is multiplied by 3. Does the first row necessarily have to contain all zeros? Explain.
23. Define the terms eigenvector and eigenvalue for a square matrix  $A$ .
24. Find the eigenvalues and eigenspaces of  $\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$ .
25. What does it mean for two matrices to be similar?
26. Diagonalize the matrix  $\begin{bmatrix} -12 & 10 \\ -15 & 13 \end{bmatrix}$ , if possible. If not say why it cannot be diagonalized.
27. Diagonalize the matrix  $\begin{bmatrix} 0 & -4 \\ 1 & 4 \end{bmatrix}$ , if possible. If not say why it cannot be diagonalized.
28. Give an example of two matrices that have the same characteristic polynomial, but they are not similar.
29. Solve the finite difference equation  $x_n = 2x_{n-1} - x_{n-2}$ ,  $x_0 = 0$ ,  $x_1 = 1$ .
30. Solve the finite difference equation  $x_n = 2x_{n-1} - 2x_{n-2}$ ,  $x_1 = 1$ ,  $x_2 = 1$ .
31. Let  $T : \mathbf{P}^3 \rightarrow \mathbf{P}^2$  be given by  $T(f) = f'$ . Use the basis  $\{1, 1+x, x+x^2, x^2+x^3\}$  for  $\mathbf{P}^3$  and the basis  $\{1, x, x^2\}$  for  $\mathbf{P}^2$  to write the matrix for  $T$ .
32. Let  $T : \mathbf{P}^3 \rightarrow \mathbf{P}^2$  be given by  $T(f) = f'$ . Use the basis  $\{1, x, x^2, x^3\}$  for  $\mathbf{P}^3$  and the basis  $\{1, 1+x, x+x^2\}$  for  $\mathbf{P}^2$  to write the matrix for  $T$ .
33. Let  $T : \mathbf{P}^3 \rightarrow \mathbf{P}^2$  be given by  $T(f) = f'$ . Use the basis  $\{1, 1+x, x+x^2, x+x^3\}$  for  $\mathbf{P}^3$  and the basis  $\{1, 1+x, x+x^2\}$  for  $\mathbf{P}^2$  to write the matrix for  $T$ .
34. Let  $T : \mathbf{P}^3 \rightarrow \mathbf{R}$  be given by  $T(f) = \int_0^1 f(x) dx$ . Use the basis  $\{1, x, x^2, x^3\}$  for  $\mathbf{P}^3$  and  $\{1\}$  for the basis of  $\mathbf{R}$  to write the matrix for  $T$ . (Here we are thinking of  $\mathbf{R}$  as a vector space.)
35. Show that  $\{3, 1+x, 2x+x^2\}$  is a basis for  $\mathbf{P}^2$ .
36. Prove that in any vector space  $V$ ,  $a\mathbf{0} = \mathbf{0}$  for any scalar  $a$ .
37. Prove that in any vector space  $V$ ,  $0\mathbf{v} = \mathbf{0}$  for any  $\mathbf{v} \in V$ .
38. Prove that in any vector space  $V$ ,  $(-1)\mathbf{v} = -\mathbf{v}$  for all  $\mathbf{v} \in V$ .
39. Prove that in any vector space  $V$ ,  $-(-\mathbf{v}) = \mathbf{v}$  for any  $\mathbf{v} \in V$ .