Definition of a Function

DEFINITION: Let A and B be sets. A function between A and B is a subset of $A \times B$ with the property that if (a, b_1) and (a, b_2) are both in the subset, then $b_1 = b_2$. The domain of the function is the set of all first coordinates in the set and the range is the set of all second coordinates in the set. The set B is sometimes called the codomain.

If it happens that the domain of f is equal to all of A, then we say that f is a function from A to B and we write $f : A \to B$.

Types of Functional Relationships

DEFINITION: A function is called surjective (or is said to map A onto B) if $B =$ range f. The question of whether or not a function is surjective depends on the choice of the set B. A function can always be made surjective by restricting the codomain to being equal to the range although this is not always convenient.

DEFINITION: A function is called injective (or one-to-one) if it satisfies the property that for all $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$.

If a function is both surjective and injective, then it is particularly well behaved.

DEFINITION: A function $f : A \to B$ is called bijective if it is both surjective and injective.

EXAMPLE: Consider the function given by the formula $f(x) = x^2$. First suppose that we take $\mathbf{R}$ (the real numbers) for both the domain and codomain so that $f : \mathbf{R} \to \mathbf{R}$. Then f is not surjective because there is no real number that maps onto -1. If we limit the codomain to be the set $\{y \in \mathbf{R} | y \geq 0\} = [0, \infty)$, then the function $f : \mathbf{R} \to [0, \infty)$ is surjective.

Since $f(1) = f(-1)$, we see that f is not injective when defined on all of $\mathbf{R}$. But by restricting f to be defined only on $[0, \infty)$, it becomes injective. Thus $f : [0, \infty) \to [0, \infty)$ is bijective.

Inverse of a Function

DEFINITION: If $y = f(x)$, then the inverse of f has the equation $x = f(y)$.

Symbol: If $y = f(x)$, then $f^{-1}(y) = x$.

Definition: If f^{-1} is a function, then f is said to be invertible.

Property: If f is invertible, then $f^{-1}(f(x)) = x$ and $f(f^{-1}(x) = x$.

Exploration: Trigonometric Functions

For the following trigonometric functions, find the domain and range.

1. $\sin x$

2. $\cos x$

3. $\tan x$

Describe a "natural" domain and range that would make each function bijective and state the inverse function.

4. $\sin x$

5. $\cos x$

6. $\tan x$

Next, consider alternate domains and ranges for the sin and cos functions.

7. Give an alternate domain and range for sin.

8. Give an alternate domain and range for cos.

9. List advantages and disadvantages of using the "natural" domain and range for sin and cos.

10. Discuss the validity of the statement "In general, the choice of domain and range used to find the inverse function is arbitrary as long as you make the function bijective." Explain in what sense the statement is true and in what sense the statement is false.

11. Think of two different functions that are commonly used in the high school mathematics curriculum whose inverses are also commonly used. Give the functions, their range and domain and their inverse functions. Don't use the inverse of part a) as your answer to part b).

    a)

    b)

For the next two parts, give the domain and range for the function. Then restrict the domain and range (as necessary) to make the function bijective and find a formula for the inverse function.

12. Let $f : \mathbf{R} - \{0\} \to \mathbf{R}$ be given by $f(x) = \frac{1}{x}$.

13. Let $g : \mathbf{R} \to \mathbf{R}$ be given by $g(x) = x^2 + 4x - 5$