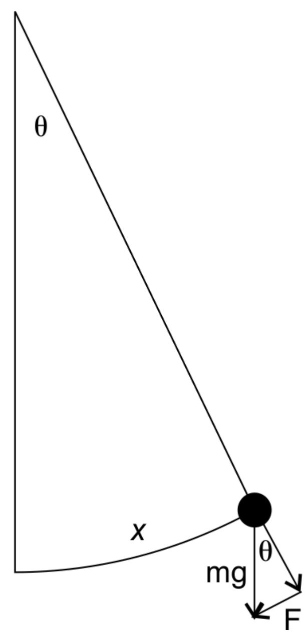
Math 1710 Project - Making a Clock

The Story. Joe wishes to design a Grandfather Clock. A Grandfather clock has a pendulum that swings back and forth at a regular rate which is the mechanism that keeps the clock accurate. He plans to make the pendulum from a heavy object that is supported by a very light wire. Joe wishes to have the period of the pendulum to be 2 seconds because that seems like a "grandfatherly" rate - not too fast, but still moving along steadily. Your job is to complete this activity and in the end, find how long the pendulum should be.

The problem. You are to analyze the geometry and physics of the situation in order to find a differential equation to model the motion of the pendulum. Then you will find a solution to the equation and determine the length of the pendulum which will make the pendulum swing a full cycle in two seconds. A well known concept in physics is that the net force on an object accelerates the object by the equation $F = ma$ where F is the force, m is the mass, and a is the acceleration. This is a differential equation since $a = x''(t)$ where $x(t)$ measures the position of the object at time t. In the case of a pendulum, the mass follows a circular path. Let x represent the distance along the circle from the equilibrium point (position when there is no motion) with x positive to the right and x negative to the left.

Procedure. There may be several ways to approach the problem, but you are asked to follow the steps outlined below.

1. Use the following drawing and trigonometry to resolve the force F into a component tangent to the circle and a component perpendicular to the circle. Note the the force of gravity is mg, so the force F in the picture is the force experienced by the object along its path of motion. (The component of the force perpendicular to the path is canceled by the force the support wire exerts on the mass.)



2. Assume as usual that the positive direction is to the right and the negative direction is to the left. Should the sign of your answer in part 1) be positive or negative?

3. In class, we proved that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Explain why it is reasonable to replace $\sin\theta$ with θ as long as the pendulum does not swing very far. Using this replace-

ment, write the equation that $F = ma$ gives in this situation. Replace θ with an expression involving the length of the pendulum h and the distance that the object is from its rest position x. Your answer is a differential equation.

4. Can you find a function of the form $x(t) = A\cos(Bt + C)$ that satisfies the differential equation? If so, what are the required conditions on the constants A, B, or C to make it work?

5. What is the period of $A\cos(Bt + C)$? (That is, what is the minimum number p such that $A\cos(Bt+C) = A\cos(B(t+p)+C)$ for all t?)

6. Based on your above calculations, how long should the pendulum be?