The Story. A bug lands on the edge of a bicycle wheel just as the bicycle starts to move. Each time the wheel goes around, the bug starts at the bottom of the wheel and hangs on tight while it goes forward with the bicycle as well as up and down. Your task is to determine the length of the path the bug travels when the bicycle travels 5 miles.

Some Background. Suppose the radius of the wheel is r. With this assumption, the path of the bug satisfies $x(t) = rt - r\sin t$ and $y(t) = r - r\cos t$ where the origin is the position at the bottom of the wheel where the bug starts, x measures the position along the road and y measures the height of the bug above the road. In class we saw the length of a path is $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$ if the path follows the function $y = f(x)$. Here the situation is slightly different. Instead of a function defining the position, the position is defined by two equations (called parametric equations), one for x and one for y, to give the position at any time t. In this case, the formula for path length is slightly different. The parametric form of the equation for path length is $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

The Problem. You are to find the length of the path the bug follows when it travels around the wheel exactly once. After you determine this, find the distance the bicycle travels when its wheel rotates exactly once. From this information, determine the length of the path the bug traveled when the bicycle travels 5 miles.

Procedure. The following is how you are to proceed to solve the problem.

1. Use the parametric formula for path length to write an integral that will give the path length for one rotation of the wheel.
2. Simplify the integral you found in 1).
3. First approximate the integral using Riemann sums with $n = 8$ and then $n = 16$. You can use a calculator or a computer, but show in your write up, what you calculuated and give the answer. If you use a computer, include the print out. Based on the Riemann sums, what do you think the actual path length is? Use Wolfram Alpha or an other system to compute the integral. Is it close to the Riemann sums you calculuated?

4. Next, compute the integral without the help of the computer. You have a square root in the integral. Use trigonometric identities to get rid of the square root. (Hint: The formula for $\cos 2\theta$ has three forms. One is $\cos 2\theta = \cos^2\theta - \sin^2\theta$. The other two can be obtained from the first by replacing either $\sin^2\theta$ with $1 - \cos^2\theta$ or replacing $\cos^2\theta$ with $1 - \sin^2\theta$. One of these should be helpful.)

5. Now integrate to obtain the path length. Of course, you should get the same answer as you obtained when you used technology.

6. Determine the ratio of the distance the bug travels divided by the distance the bicycle travels forward by computing this ratio when the wheel goes around once.

7. Use your ratio to determine the approximate length of the path the bug travels when the bicycle travels 5 miles.