3) If $f: A \to \mathbb{R}$ is continuous, then for any closed set $C \subseteq \mathbb{R}$, $f^{-1}(C) = A \cap K$ for some closed set K.

Pf:

Since $C \subseteq \mathbb{R}$ is closed, $\mathbb{R} - C$ is open. Therefore by lemma proved in class, $\exists$ open set $U \ni$ $f^{-1}(\mathbb{R} - C) = A \cap U$.

By Theorem 2.3.16 in the book,

$$\begin{aligned} f^{-1}(C) &= A - f^{-1}(\mathbb{R} - C) \\ &= A - (A \cap U) \\ &= A - U \\ &= (\mathbb{R} - U) \cap A \\ &= A \cap (\mathbb{R} - U) \end{aligned}$$

Let $K = \mathbb{R} - U$. K is closed since U is open.

$\therefore f^{-1}(C) = A \cap K$ for some closed set K.

4) If $\emptyset \neq A \subseteq [a,b]$, $A = [a,b] \cap K$ and $A = [a,b] \cap V$ where K is closed and V is open, then $A = [a,b]$.

pf: Let $S = \{t \in [a,b] \mid [a,t] \subseteq A\}$.

$S \neq \emptyset$ since $a \in S$ and $S \subseteq [a,b]$, so S is bounded above by b.

$\therefore$ sup S exists.

Let $r = \sup S$.

We first show $r \in A$.

Let $\varepsilon > 0$ be arbitrary. Then $\exists x \in (r-\varepsilon, r] \ni x \in S$ since $r = \sup S$. $\therefore$ $[a,x] \subseteq A$ $\therefore$ $N^*(r;\varepsilon) \cap A \neq \emptyset$.

$\therefore r \in A'$.

Since $A = [a,b] \cap K$ is closed, $A' \subseteq A$. $\therefore r \in A$.

We next show $r \in S$.

Since $a \in S$, $a \leq r = \sup S$. Since b is an upper bound for S, $r \leq b$. $\therefore$ $r \in [a,b]$.

Let $x \in [a,r]$ if $x = r$, $x \in A$ from the above.

if $x < r$ then $\exists$ $y \in (x, r] \ni y \in S$ since $r = \sup S$.

$\therefore [a,x] \subseteq [a,y] \subseteq A$. $\therefore$ $x \in A$.

$\therefore [a,r] \subseteq A$ $\therefore r \in S$

Now we show that $r = b$ using proof by contradiction.

Suppose $b \neq r$. Then $r < b$.

Since $r \in A = [a,b] \cap U$, $r \in U$. So r is an interior pt of U since U is open.

$\therefore \exists\, \varepsilon > 0 \ni N(r;\varepsilon) \subseteq U$.

Let $w = \min\{r + \frac{\varepsilon}{2}, \frac{b+r}{2}\}$. Note that $w > r$, $w \in N(r;\varepsilon)$, and $w < b$.

Then $[r,w] \subseteq N(r;\varepsilon) \subseteq U$ and $[r,w] \subseteq [a,b]$.

$\therefore [r,w] \subseteq [a,b] \cap U = A$.

But $[a,r] \subseteq A$ since $r \in S$.

$\therefore [a,w] = [a,r] \cup [r,w] \subseteq A$
and $w \in [a,b]$.
$\therefore w \in S$.
But $r < w$ and $r = \sup S$. #

$\therefore r = b$.

$\therefore b \in S \quad \therefore [a,b] \subseteq A$.

Since $A \subseteq [a,b]$, $A = [a,b]$.