Math 3000 Homework due November 18

1. Prove that if $A \subseteq \mathbf{R}$ and $f, g : A \to \mathbf{R}$ are both continuous, then
   a. For any $c \in \mathbf{R}$, the function $w : A \to \mathbf{R}$ given by $w(x) = cf(x)$ is also continous. (Hint: if $\epsilon > 0$, then $\epsilon' = \frac{\epsilon}{k} > 0$ for any positive value of k. Choose a value for k wisely.)
   b. The function $s : A \to \mathbf{R}$ given by $s(x) = f(x) + g(x)$ is continouous. (Hint: if $\epsilon > 0$, then $\frac{\epsilon}{2} > 0$.)
   c. The function $p : A \to \mathbf{R}$ given by $p(x) = f(x)g(x)$ is continous. (Hint: $|f(x)g(x) - f(a)g(a)| = |f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)|$.)

2. Prove that if $C \subseteq \mathbf{R}$ is compact and $C \neq \emptyset$, then $\sup(C) \in C$.

3. Prove that if $f : A \to \mathbf{R}$ is continuous, then for any closed set $C \subseteq \mathbf{R}$, $f^{-1}(C) = K \cap A$ for some closed set $K \subseteq \mathbf{R}$. (Hint: Use a similar fact about open sets proved in class and Theorem 2.3.16 in the book.)

4. Suppose that $A \subseteq [a, b]$, $a \in A$, and $A = [a, b] \cap K = [a, b] \cap V$ where $K \subseteq \mathbf{R}$ is closed and $V \subseteq \mathbf{R}$ is open. Prove that $A = [a, b]$. (Hint: Let $S = \{x \in [a, b] \,|\, \text{if } t \in [a, b] \text{ and } t \leq x, \text{ then } t \in A\}$. Then do pretty much what we did to prove the Intermediate Value Theorem.)