MATH 3000 Final Exam May 6, 2013 Name____________________________________

You are not to use a calculator, book, notes, or others to do this exam. **Show all essential work.**

1. Let $A \subseteq \mathbf{R}$, $x \in \mathbf{R}$ and $f : A \to R$. Give the definition of each of the following. *(50 points)*

   a. x is an interior point for the set A.

   b. $f^{-1}(D)$ where $D \subseteq \mathbf{R}$.

   c. A is a closed set.

   d. A is an open set.

   e. f is injective.

   f. A is compact.

   g. f is continuous.

   h. x is an accumulation point of A.

2. Negate the following statements and then say which is true, the original statement, the negation, neither, or both. *(10 points)*

a. $\forall x \in \mathbf{N}, \exists x \in \mathbf{R} \ni 0 < x < \frac{1}{n}$

b. $\exists x \in \mathbf{R} \ni \forall y \in \mathbf{R}, x^2 < 0 \Rightarrow y^2 < 0$

3. Use the definitions of A' and $\mathrm{bd}(A)$ to prove that for any subset $A \subseteq \mathbf{R}$, $A' \subseteq A \cup \mathrm{bd}(A)$. *(15 points)*

4. Which of the following are compact? *(10 points)*

   a. $\{0\} \cup \left\{ \dfrac{1}{n} : n \in \mathbf{N} \right\}$

   b. $[0, 1] \cap (\mathbf{R} - \mathbf{Q})$

   c. $\mathbf{Z}$

5. State each of the following. Be sure to give a precise and correct statement. *(20 points)*

   a. The Completion Axiom

   b. Heine-Borel Theorem

   c. Bolzano Weierstrass Theorem

   d. Intermediate Value Thoerem

6. For each of the sets, give an example of an open cover with no finite subcover. You do not need to prove your answer, just carefully state the open cover you wish to use for your example. *(10 points)*

   a. $(0, 1]$

   b. $[0, \infty)$

7. Use induction to prove that for any natural number $n \geq 2$, $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{(n-1) \cdot n} = \dfrac{n-1}{n}$. *(15 points)*

8. Prove that if $A_\alpha \subseteq \mathbf{R}$ is open for each $\alpha \in I$, then $\bigcup_{\alpha \in I} A_\alpha$ is an open set. *(10 points)*

9. Use the definition of continuous to prove that if $f : \mathbf{R} \to \mathbf{R}$ is given by $f(x) = \dfrac{10}{x+3}$, then f is continous at 4. *(10 points)*

10. Use the intermediate value theorem to prove that there is a real number whose square is 2. *(10 points)*

11. True or false. If true just state that it is true. If false, state it is false and then **give a counter example.** *(35 points)*

a. If $A \subseteq \mathbf{R}$ is not open, then a is closed.

b. Let $S \subset \mathbf{R}$. Then $\mathrm{bd}S \subseteq S'$.

c. If $A \subset B$ and $A \neq B$, then A and B do not have the same cardinality.

d. If $f : A \to B$, $D \subset B$, then $f(f^{-1}(D)) = D$.

e. If $A \subseteq \mathbf{R}$ is bounded and $f : A \to \mathbf{R}$ is continuous, then $f(A)$ is bounded.

f. If $A \subseteq \mathbf{R}$ is both closed and bounded and $f : \mathbf{R} \to \mathbf{R}$ is continuous, then $f(A)$ is closed.

g. If $B \subseteq \mathbf{R}$ is open and $f : \mathbf{R} \to \mathbf{R}$ is continuous, then $f^{-1}(B)$ is open.

$11.\overline{9}$ Have a good summer break! *(10 points)*