MATH 3000 Exam 1 September 30, 2013 Name________________________

You are not to use a calculator, book, notes, or others to do this exam.

1. Define the following. *(30 points)*

   a. $N(x;\epsilon)$ where $x \in \mathbf{R}$ and $\epsilon > 0$

   b. x is a boundary point of the set $S \subseteq \mathbf{R}$

   c. x is an interior point of the set $S \subseteq \mathbf{R}$

   d. $S \subseteq \mathbf{R}$ is an open set

   e. $S \subseteq \mathbf{R}$ is a closed set

   f. Let A and B be sets. $A - B$

   g. $\bigcup_{\alpha \in I} A_\alpha$ where for each $\alpha \in I$, A_α is a set.

In your notes and book.

2. Give the truth table for $(p \Rightarrow q) \wedge \sim q$. *(5 points)*

p	q	$(p \Rightarrow q) \wedge \sim q$
T	T	F
T	F	F
F	T	F
F	F	T

3. Negate the following statements. For each part indicate if the original statement is true and if the negated statement is true. *(15 points)*

a. $\forall x \in \mathbf{R}, x^2 + 4x + 3 > 0$

$\exists x \in \mathbb{R} \ni x^2 + 4x + 3 \leq 0$

Statement true? No

Negation true? Yes $(-2)^2 + 4(-2) + 3 = -1$

b. $\forall x \in \mathbf{R}, \exists y \in \mathbf{R} \ni (x > 0) \Rightarrow (y^2 \leq x)$

$\exists x \in \mathbb{R} \ni \forall y \in \mathbb{R}, \ x > 0 \wedge y^2 > x$

Statement true? Yes

Let $y = \begin{cases} 0 & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$

Negation true? No

c. $\forall x \in \mathbf{R}, \exists n \in \mathbf{N} \ni (x \leq 0) \vee (\frac{1}{n} < x)$

$\exists x \in \mathbb{R} \ni \forall n \in \mathbb{N}, (x > 0) \wedge (\frac{1}{n} \geq x)$

Statement true? Yes

Negation true? No

This is Archimedian Property.

4. Let $S = \{ \frac{1}{n} | n \in \mathbf{N}\} \cup [3,5)$ *(10 points)*

a. Find bd(S)

$bd(S) = \{0\} \cup \{ \frac{1}{n} | n \in \mathbb{N}\} \cup \{3,5\}$.

b. Find int(S) $= (3,5)$

5. Prove that for any $A, B \subset \mathbf{R}$, $\mathbf{R} - (A \cup B) = (\mathbf{R} - A) \cap (\mathbf{R} - B)$. *(10 points)*

We first show $\mathbb{R} \setminus (A \cup B) \subseteq (\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B)$.

Let $x \in \mathbb{R} \setminus (A \cup B)$. $\therefore$ $x \in \mathbb{R}$ and $x \notin A \cup B$.

Since $x \notin A \cup B$, the statement $x \in A$ or $x \in B$ is false.

$\therefore$ $x \notin A$ and $x \notin B$. Since $x \in \mathbb{R}$ and $x \notin A$, $x \in \mathbb{R} \setminus A$.

Since $x \in \mathbb{R}$ and $x \notin B$, $x \in \mathbb{R} \setminus B$.

$\therefore$ $x \in (\mathbb{R} - A) \cap (\mathbb{R} \setminus B)$, so $\mathbb{R} \setminus (A \cup B) \subseteq (\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B)$.

Next we show $(\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B) \subseteq \mathbb{R} \setminus (A \cup B)$.

Let $x \in (\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B)$.

$\therefore$ $x \in \mathbb{R} - A$ and $x \in \mathbb{R} - B$.

$\therefore$ $x \in \mathbb{R}$, $x \notin A$ and $x \notin B$.

$\therefore$ $x \notin A \cup B$

$\therefore$ $x \in \mathbb{R} \setminus (A \cup B)$

$\therefore$ $(\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B) \subseteq \mathbb{R} \setminus (A \cup B)$.

$\therefore$ $(\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B) = \mathbb{R} \setminus (A \cup B)$.

6. Use the definition of boundary point to prove that 0 is a boundary point of $(0, 1)$. *(10 points)*

Let $\varepsilon > 0$ be arbitrary. Then $-\varepsilon/2 < 0$, ~~so~~ $-\varepsilon/2 \in \mathbb{R} \setminus (0,1)$.
Also $-\varepsilon/2 \in N(0;\varepsilon)$ since $|0 - (-\varepsilon/2)| = \varepsilon/2 < \varepsilon$.
$\therefore N(0;\varepsilon) \cap (\mathbb{R} \setminus (0,1)) \neq \emptyset$.

Next, let $x = \min\{\frac{\varepsilon}{2}, \frac{1}{2}\}$. Then $|0 - x| = x \leq \frac{\varepsilon}{2}$, so $x \in N(0;$
also $0 < x \leq \frac{1}{2}$, so $x \in (0,1)$.
$\therefore N(0;\varepsilon) \cap (0,1) \neq \emptyset$.
$\therefore 0 \in \text{bd}(0,1)$.

7. State whether each statement is true or false. If true, just state that it is true. If false, say it is false and then find a counterexample that shows it is false. You need not give explanation with your counterexamples. Keep in mind that I am asking if the statement is true, and not if it is the full definition of something. *(15 points)*

a. Any union of open sets is open.

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b. Any union of closed sets is closed.

F $\quad [0,1] = \bigcup_{x \in [0,1]} \{x\}$

c. If $A, B \subseteq \mathbf{R}$, then $\text{bd}(A \cup B) = \text{bd}(A) \cup \text{bd}(B)$.

T $\quad A = (-\infty, 0]$, $B = [0, \infty)$
$\text{bd}(A) = \{0\}$ $\quad \text{bd}(B) = \{0\}$
$A \cup B = \mathbb{R}$ and $\text{bd}(\mathbb{R}) = \emptyset$.

d. A set $S \subseteq \mathbf{R}$ is open if and only if $\text{int}(S) = S$.

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e. A set $S \subseteq \mathbf{R}$ is closed if and only if $\text{bd}(S) = S$.

F $\quad [0,1]$ closed, but $\text{bd}[0,1] = \{0,1\}$.

MATH 3000.001 Exam 2 Oct 28, 2013 Name______________________

You are not to use a calculator, book, notes, or others to do this exam.

1. Define the following. *(25 points)*

   a. An accumulation point of a a set $S \subseteq \mathbf{R}$.

   b. The closure of a set $S \subseteq \mathbf{R}$.

   In book & notes.

   c. $\sup(S)$ where $S \subseteq \mathbf{R}$.

   d. For $f : A \to B$, define what it means for f to be surjective.

   e. For $f : A \to B$ and $C \subseteq A$, define $f(C)$.

2. State the completion axiom. *(5 points)*

3. Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^2$. Find the following sets. *(6 points)*

a. $f([-1, 2))$

$= [0,4)$

c. $f^{-1}([16, 25))$ $= [4,5) \cup (-5,-4]$

4. Find the following sets. *(10 points)*

a. $\mathrm{cl}(\mathbf{R} - \mathbf{Q})$ $= \mathbb{R}$

c. $((0,1] \cap \mathbf{Q})'$ (The set of accumulation points of the rational numbers in the interval $(0,1]$.)

$[0,1]$

5. Give the maximum, minimum, supremum and infimum of each set, if they exist. State for which they do not exist.

a. $\mathbf{N}$, the natural numbers. *(8 points)*

Maximum = DNE Minimum = 1

Supremum = DNE Infimum = 1

b. $\{x \in \mathbf{Q} \mid x^2 < 2\}$

Maximum = DNE Minimum = DNE

Supremum = $\sqrt{2}$ Infimum = $-\sqrt{2}$

6. Prove that if $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f : A \to C$ is also injective. *(10 points)*

Suppose $g \circ f(x_1) = g \circ f(x_2)$ for $x_1, x_2 \in A$.

Then $g(f(x_1)) = g(f(x_2))$

$\therefore$ $f(x_1) = f(x_2)$ since g is injective

$\therefore$ $x_1 = x_2$ since f is injective

$\therefore$ $g \circ f$ is injective.

7. Prove that if $S \subseteq \mathbf{R}$, then $\mathrm{bd}(S) \subseteq S \cup S'$. *(10 points)*

Let $x \in \mathrm{bd}(S)$.

If $x \in S$, then $x \in S \cup S'$.

So we assume $x \notin S$ and show $x \in S'$.

Let $\varepsilon > 0$ be arbitrary.

Since $x \in \mathrm{bd}(S)$, $\exists\, y \in N(x;\varepsilon) \cap S$.

$y \neq x$ since $y \in S$ and $x \notin S$.

$\therefore$ $y \in N^*(x;\varepsilon) \cap S$.

$\therefore$ $N^*(x;\varepsilon) \cap S \neq \phi$.

Since $\varepsilon > 0$ is arbitrary, $x \in S'$.

8. Prove that if $S \subseteq \mathbf{R}$, $\sup(S)$ exists and S is open, then $\sup(S) \notin S$). *(10 points)*

Use proof by contradiction. Suppose $\sup S \in S$.

Since S is open, $\sup S$ is an interior point of S.

$\therefore \exists \varepsilon > 0 \ni N(\sup S; \varepsilon) \subseteq S$. But $\sup(S) + \frac{\varepsilon}{2} \in N(\sup S; \varepsilon)$

so $\sup(S) + \frac{\varepsilon}{2} \in S$. This contradicts the fact that

$\sup(S)$ is an upper bound for S. ※

$\therefore \sup S \notin S$.

9. State whether each statement is true or false. If true, just state that it is true. If false, say it is false and then find a counterexample that shows it is false. You need not give explanation with your counterexamples. Keep in mind that I am asking if the statement is true, and not if it is the full definition of something. *(20 points)*

a. Every nonempty open subset of the real numbers contains an irrational number.

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b. If $f : A \to B$ and $C \subseteq A$, then $C = f^{-1}(f(C))$.

F $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

$C = [0,1]$, $f^{-1}(f(C)) = f^{-1}([0,1]) = [-1,1]$.

c. For any $S \subseteq \mathbf{R}$, if x is a boundary point of S, then x is an accumulation point of S.

F $S = \{0\}$ $0 \in \operatorname{bd}(S)$, $0 \notin S'$

d. If $S \subseteq \mathbf{R}$, $\sup(S) \in S$ and $\inf(S) \in S$, then S is closed.

F $[0,1) \cup (2,3]$

e. A set $S \subseteq \mathbf{R}$ is closed if and only if $S' \subseteq S$.

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MATH 3000.001 Exam 3 Nov 25, 2013 Name____

You are not to use a calculator, book, notes, or others to do this exam.

In book and notes.

1. Define the following. *(15 points)*

   a. Suppose that $f : A \to \mathbf{R}$ and $a \in A \subseteq \mathbf{R}$. Give the definition of f is continuous at a.

   b. An open cover for a set $S \subseteq \mathbf{R}$. (Be sure to include in your definition both open and cover.)

   c. $S \subseteq \mathbf{R}$ is a compact set.

2. State the following theorems. *(20 points)*

   a. Heine-Borel Theorem

   b. Bolzano-Weierstrass Theorem

   c. Intermediate Value Theorem

   d. Extreme Value Theorem

3. Which of the following sets are compact? Give the reason that you know the set is or is not compact. *(20 points)*

a. $[0, 100]$ compact - closed + bounded

b. $[0, \infty)$ Not compact - not bounded

c. $\displaystyle\bigcup_{n\in\mathbb{N}} \left[\frac{1}{n}, 2\right] = (0,2]$ not compact - not closed.

d. $\displaystyle\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, 2+\frac{1}{n}\right] = [0,2]$ compact - closed and bounded.

4. Let $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. **Use the definition** of compact to show that S is not compact. (If you use a theorem instead of the definition, then will get little or no credit.) *(10 points)*

Let $\mathcal{Q} = \{(\frac{1}{2}, \infty)\} \cup \{(\frac{1}{n+1}, \frac{1}{n-1}) \mid n \in \mathbb{N}\}$.

Note that for each $n \in \mathbb{N}$, n is in exactly one set from the collection $\mathcal{Q}$. Namely, $1 \in (\frac{1}{2}, \infty)$ and for $n \geq 2$, $n \in (\frac{1}{n+1}, \frac{1}{n-1})$. Also, each set in $\mathcal{Q}$ contains a natural number.

$\therefore$ No proper subset of $\mathcal{Q}$ is a cover, but $\mathcal{Q}$ is an open cover for S.

$\therefore$ $\mathcal{Q}$ contains no finite subcover for S.

$\therefore$ S is not compact.

5. Use the definition of continuous to prove that if $f : \mathbf{R} \to \mathbf{R}$ is given by $f(x) = 3x^2 + 2x - 1$, then f is continuous at 2. *(10 points)*

Let $\varepsilon > 0$ be arbitrary. Let $\delta = \min\{1, \varepsilon/17\}$.

$\delta > 0$ since $1 > 0$ and $\varepsilon/17 > 0$. Suppose that $x \in \mathbb{R}$

and $|x-2| < \delta$.

$\therefore |x-2| < 1$

$\therefore -1 < x-2 < 1$

$\therefore 1 < x < 3$

$\therefore 3 < 3x < 9$

$\therefore 9 < 3x+8 < 17$

also,
$$\begin{aligned} |f(x) - f(2)| &= |3x^2+2x-1-(3\cdot 2^2+2\cdot 2-1)| \\ &= |3x^2+2x-16| \\ &= |x-2||3x+8| \\ &\leq |x-2| \cdot 17 \\ &< \delta \cdot 17 \\ &\leq \tfrac{\varepsilon}{17} \cdot 17 \\ &= \varepsilon. \end{aligned}$$

6. Prove that if $A \subseteq \mathbf{R}$ and $f, g : A \to \mathbf{R}$ are both continuous, then the function $s : A \to \mathbf{R}$ given by $s(x) = f(x) + g(x)$ is continuous. *(10 points)*

Let $a \in A$.

Let $\varepsilon > 0$ be arbitrary. Then $\varepsilon/2 > 0$.

Since f is continuous at a, $\exists\, \delta_1 > 0 \ni$ if $|x-a| < \delta_1$, then $|f(x) - f(a)| < \varepsilon/2$

Since g is continuous at a, $\exists\, \delta_2 > 0 \ni$ if $|x-a| < \delta_2$, then $|g(x) - g(a)| < \varepsilon/2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. $\delta > 0$ since $\delta_1 > 0$ & $\delta_2 > 0$.

Let $x \in A$ with $|x-a| < \delta$.

$\therefore |x-a| < \delta_1$ which implies $|f(x) - f(a)| < \varepsilon/2$

also $|x-a| < \delta_2$ which implies $|g(x) - g(a)| < \varepsilon/2$

$$\begin{aligned} \therefore |s(x) - s(a)| &= |f(x) + g(x) - (f(a) + g(a))| \\ &= |f(x) - f(a) + (g(x) - g(a))| \\ &\leq |f(x) - f(a)| + |g(x) - g(a)| \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

$\therefore$ S is cont at $a \in A$. Since a was arbitr[illegible]

7. Assume that $C \subseteq \mathbf{R}$ is a closed set, $0 \in C$ and $S = \{x \in [0,1] \mid [0,x] \subseteq C\}$. *(10 points)*

a. Prove $\sup(S)$ exists.

$[0,0] \subseteq C \quad \therefore 0 \in S$

$S \subseteq [0,1]$, so S is bounded.

$\therefore \sup S$ exists by completion axiom.

b. Prove $\sup(S) \in S$. Let $\varepsilon > 0$. Then $\exists x \in (\sup S - \varepsilon, \sup S] \cap S$.

$\therefore [0,x] \subseteq C$. $\therefore x \in C \cap N(\sup S; \varepsilon)$. Also,

$N(\sup S; \varepsilon) \cap (\mathbb{R} \setminus C) \neq \emptyset$ or else $[0, \sup S + \frac{\varepsilon}{2}] \subseteq C \Rightarrow \sup S + \frac{\varepsilon}{2} \in S$ ※

$\therefore \sup S \in bd(C)$. $\therefore \sup S \in C$ since C is closed.

We next show that $[0, \sup S) \subseteq C$. Let $t \in [0, \sup S)$. There is

an $x \in S \ni x > t$ since $t < \sup S$. $\therefore [0,x] \subseteq C \therefore t \in C$.

$\therefore [0, \sup S) \subseteq C$. Since $\sup S \in C$, $[0, \sup S] \subseteq C$.

$\therefore \sup S \in S$.

8. State whether each statement is true or false. If true, just state that it is true. If false, say it is false and then find a counterexample that shows it is false. You need not give explanation with your counterexamples. Keep in mind that I am asking if the statement is true, and not if it is the full definition of something. *(15 points)*

a. The union of two compact subsets of $\mathbf{R}$ is compact.

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b. If $f : \mathbf{R} \to \mathbf{R}$ is continuous and $C \subseteq \mathbf{R}$ is closed, then $f(C)$ is closed.

F $\quad f(x) = \frac{1}{1+x^2} \quad f([0,\infty)) = (0,1]$

c. If $f : \mathbf{R} \to \mathbf{R}$ is continuous and $C \subseteq \mathbf{R}$ is both closed and bounded, then $f(C)$ is closed and bounded.

T

d. If $A \subseteq \mathbf{R}$, $f : A \to \mathbf{R}$ is continuous and $U \subseteq \mathbf{R}$ is open, then $f^{-1}(U)$ is open.

F $\quad A = [0,1]$, $f(x) = x$, $U = (\frac{1}{2}, 10)$, then $f^{-1}(U) = (\frac{1}{2}, 1]$.