Math 2510 Fill in the reasons

a) If $x+z=y+z$, then $x=y$.

Pf: $x+z=y+z$

$$\therefore (x+z)+(-z)=(y+z)+(-z)$$
$$x+(z+-z)=y+(z+-z)$$
$$x+0=y+0$$
$$x=y.$$

$\square$

b) If $x\in\mathbb{R}$, then $x\cdot 0=0$.

Pf:
$$\begin{aligned} x\cdot 0+x\cdot 0 &= x(0+0)\\ &= x\cdot 0\\ &= x\cdot 0+0\\ &= 0+x\cdot 0 \end{aligned}$$

$$\therefore x\cdot 0=0$$

$\square$

c) If $x\in\mathbb{R}$, then $(-1)x=-x$

Pf:
$$\begin{aligned} (-1)x+x &= (-1)x+1\cdot x\\ &= (-1+1)x\\ &= 0x\\ &= 0 \end{aligned}$$

$\square$

d) If $x,y\in\mathbb{R}$ and $xy=0$, then $x=0$ or $y=0$

Pf: We assume $xy=0$. If $x=0$, we are done. So we assume $x\neq 0$. Then

$$\begin{aligned} xy&=0\\ (xy)\tfrac{1}{x}&=0\cdot\tfrac{1}{x}\\ &=\tfrac{1}{x}\cdot 0\\ &=0. \end{aligned}$$

$\therefore\ 0 = (xy)\frac{1}{x}$

$= x(y \cdot \frac{1}{x})$

$= x(\frac{1}{x} y)$

$= (x \cdot \frac{1}{x}) y$

$= 1 y$

$= y$

$\therefore\ y = 0$ $\square$

e) $(-1)(-1) = 1$

Pf: $(-1)(-1) + (-1)(1) = -1(-1+1)$

$= -1(0)$

$= 0 \cdot (-1)$

$= 0$

$\therefore\ (-1)(-1) + -1 = 0$

$\therefore\ (-1)(-1) + -1 = 1 + -1$

$\therefore\ (-1)(-1) = 1.$ $\square$

f) $1 > 0$

Pf: Either $1 = 0$, $1 < 0$, or $1 > 0$.

$1 \neq 0$

$\therefore\ 1 < 0$ or $1 > 0$

If $1 < 0$, then $1 + (-1) < 0 + -1$

$\therefore\ 0 < -1 + 0$

$\therefore\ 0 < -1$

$\therefore\ 0(-1) < (-1)(-1)$

$\therefore\ (-1)0 < (-1)(-1)$

$0 < (-1)(-1)$

$0 < 1$

$\therefore\ 1 > 0.$ $\square$

g) If $x \in \mathbb{R}$, then $x > 0$ iff $-x < 0$.

~~Pf~~: $(\Rightarrow)$ Suppose $x > 0$.

$$x + (-x) > -x$$

$$0 > -x$$

$$\therefore\ -x < 0.$$

$(\Leftarrow)$ Suppose $-x < 0$.

$$-x + x < 0 + x$$

$$x + -x < 0 + x$$

$$0 < 0 + x$$

$$0 < x$$

$$\therefore\ x > 0.$$ $\square$

h) If $a, b \in \mathbb{R}$, $a \neq 0$ and $b \neq 0$, then $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$.

~~Pf~~:
$$\begin{aligned}(ab)\cdot\left(\tfrac{1}{a}\cdot\tfrac{1}{b}\right) &= \left((ab)\cdot\tfrac{1}{a}\right)\cdot\tfrac{1}{b} \\ &= \left(a\left(b\cdot\tfrac{1}{a}\right)\right)\cdot\tfrac{1}{b} \\ &= \left(\left(a\cdot\tfrac{1}{a}\right)b\right)\cdot\tfrac{1}{b} \\ &= (1\cdot b)\tfrac{1}{b} \\ &= b\cdot\tfrac{1}{b} \\ &= 1\end{aligned}$$

$ab \neq 0.$

$$\therefore\ \left((ab)\cdot\left(\tfrac{1}{a}\cdot\tfrac{1}{b}\right)\right)\tfrac{1}{ab} = \tfrac{1}{ab}$$

$$(ab)\left(\left(\tfrac{1}{a}\cdot\tfrac{1}{b}\right)\left(\tfrac{1}{ab}\right)\right) = \tfrac{1}{ab}$$

$$(ab)\left(\left(\tfrac{1}{ab}\right)\cdot\left(\tfrac{1}{a}\cdot\tfrac{1}{b}\right)\right) = \tfrac{1}{ab}$$

$$\left((ab)\left(\frac{1}{ab}\right)\right)\cdot\left(\frac{1}{a}\cdot\frac{1}{b}\right)=\frac{1}{ab}$$

$$1\cdot\left(\frac{1}{a}\cdot\frac{1}{b}\right)=\frac{1}{ab}$$

$$\left(\frac{1}{a}\cdot\frac{1}{b}\right)\cdot 1=\frac{1}{ab}$$

$$\frac{1}{a}\cdot\frac{1}{b}=\frac{1}{ab} \qquad \square$$

i) If $a,b,c,d\in\mathbb{R}$ with $b\neq 0$ and $d\neq 0$, then

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$

Pf: $\frac{a}{b}+\frac{c}{d}=a\cdot\frac{1}{b}+c\cdot\frac{1}{d}$

$$=\left(a\cdot\frac{1}{b}\right)(1)+\left(c\cdot\frac{1}{d}\right)(1)$$

$$=\left(a\cdot\frac{1}{b}\right)\left(d\cdot\frac{1}{d}\right)+\left(c\cdot\frac{1}{d}\right)\left(b\cdot\frac{1}{b}\right)$$

$$=\left(\left(a\cdot\frac{1}{b}\right)d\right)\frac{1}{d}+\left(\left(c\cdot\frac{1}{d}\right)b\right)\cdot\frac{1}{b}$$

$$=\left(a\left(\frac{1}{b}\cdot d\right)\right)\frac{1}{d}+\left(c\left(\frac{1}{d}\cdot b\right)\right)\frac{1}{b}$$

$$=\left(a\left(d\cdot\frac{1}{b}\right)\right)\frac{1}{d}+\left(c\left(b\cdot\frac{1}{d}\right)\right)\cdot\frac{1}{b}$$

$$=\left((ad)\cdot\frac{1}{b}\right)\frac{1}{d}+\left((cb)\cdot\frac{1}{d}\right)\cdot\frac{1}{b}$$

$$=(ad)\cdot\left(\frac{1}{b}\cdot\frac{1}{d}\right)+(cb)\left(\frac{1}{d}\frac{1}{b}\right)$$

$$=(ad)\cdot\left(\frac{1}{bd}\right)+(cb)\cdot\left(\frac{1}{db}\right)$$

$$=(ad)\cdot\frac{1}{bd}+(cb)\cdot\frac{1}{bd}$$

$$= \left(\frac{1}{bd}\right)\cdot(ad) + \left(\frac{1}{bd}\right)(cb)$$

$$= \frac{1}{bd}\left(ad + cb\right)$$

$$= (ad+cb)\cdot\left(\frac{1}{bd}\right)$$

$$= \frac{ad+cb}{bd}$$