MATH 2510 **Exam 3** **May 6, 2009** **Name** Key

You are not to use a calculator, book, notes, or others to do this exam. **Show all essential work.**

1. For each of the following sets, find the supremum, infimum, the boundary points, and the accumulation points. *(10 points)*

   a. $S = [0, 1)$

   $\sup(S) =$ 1 $\qquad \inf(S) =$ 0 $\qquad \mathrm{bd}(S) =$ $\{0,1\}$ $\qquad S' =$ $[0,1]$

   b. $S = \left\{ \frac{1}{n} \mid n \in \mathbf{N} \right\}$

   $\sup(S) =$ 1 $\qquad \inf(S) =$ 0 $\qquad \mathrm{bd}(S) =$ $\{0\} \cup S$ $\qquad S' =$ $\{0\}$

2. State the completion axiom. See page 121. *(5 points)*

3. Let $S \subseteq \mathbf{R}$. State the definition of the following. (Don't give an equivalent condition, give the definition.) *(10 points)*

   a. S is open.

   See page 131

   b. S is closed See page 131

4. Find the interior of the set $S = [2, 3] \cup \mathbf{Q}$. *(6 points)*

$(2,3)$

5. Find all the isolated points in the set $S = \{\frac{1}{n} \mid n \in \mathbf{N}\} \cup (1, \infty)$. *(6 points)*

$S \sim \{1\} = \{\frac{1}{n} \mid n \in \mathbb{N}, n > 1\}$

6. Find the closure of the set $S = \{\frac{1}{2^n} \mid n \in \mathbf{N}\}$. *(6 points)*

$S \cup \{0\}$

7. Prove that if S is not empty, S is closed and S is bounded, then $\sup(S) \in S$. *(15 points)*

Since $S \neq \phi$ and S is bounded, $\sup(S)$ exists. Let $x = \sup(S)$.
Since S is closed, we only need to show $x \in bd(S)$ because $bd(S) \subseteq S$.

We let $\varepsilon > 0$ be arbitrary. Then $x + \frac{\varepsilon}{2} \in N(x;\varepsilon)$ and $x + \frac{\varepsilon}{2} > x$. Since $x = \sup(S)$ is an upper bound for S, $x + \frac{\varepsilon}{2} \notin S$. $\therefore$ $x \in N(x;\varepsilon) \cap (\mathbb{R} - S)$, so $N(x;\varepsilon) \cap (\mathbb{R} - S) \neq \phi$.

To show $x \in bd(S)$ we now only need to show $N(x;\varepsilon) \cap S \neq \phi$.
Suppose that $N(x;\varepsilon) \cap S = \phi$. Then $x - \varepsilon$ would be an upper bound for S. But $x = \sup(S)$ is the least upper bound for S, which gives a contradiction since $x - \varepsilon < x$.
$\therefore$ $N(x;\varepsilon) \cap S \neq \phi$ $\quad$ $\therefore$ $x \in bd(S) \subseteq S$. $\quad$ $\therefore$ $\sup(S) \in S$.

8. Prove that if $x \in \mathrm{bd}(S) - S$, then $x \in S'$. *(15 points)*

Let $x \in bd(S) - S$. We show $x \in S'$.

Let $\varepsilon > 0$. Then $N(x;\varepsilon) \cap S \neq \phi$ since $x \in bd(S)$.

Since $x \notin S$, $(N(x;\varepsilon) - \{x\}) \cap S = N(x;\varepsilon) \cap S \neq \phi$.

$\therefore$ $N^*(x;\varepsilon) \cap S \neq \phi$.

$\therefore$ $\forall \varepsilon > 0$, $N^*(x;\varepsilon) \cap S \neq \phi$.

$\therefore$ $x \in S'$.

10. True or False? If true, then just state that it is true. If false, then state it is false and either change the statement in some nontrivial way to make it true or else give a counter example. *(28 points)*

a. For any set $S \subseteq \mathbf{R}$, $\mathrm{bd}(S) = S'$.

False

Let $S = [0,1]$. $\mathrm{bd}(S) = \{0,1\}$ and $S' = [0,1]$.

b. There is a subset of $\mathbf{R}$ which is both open and closed.

True

$\mathbb{R}$

c. For any set $S \subseteq \mathbf{R}$, S is open or S is closed.

False

$S = (0,1]$ (Note: $S = \mathbb{R}$ is not a counter example.)

d. For any nonempty set $S \subseteq \mathbf{R}$, if S has both a maximum and a minimum, then S is closed.

False

$S = [0,1) \cup (2,3]$.

e. For any subset $S \subseteq \mathbf{R}$, $\mathrm{bd}(S) = \mathrm{bd}(\mathbf{R} - S)$.

True

f. For any subset $S \subseteq \mathbf{R}$, $S' = (\mathbf{R} - S)'$.

False

$S = \{0\}$ $S' = \phi$ $(\mathbb{R} - S)' = \mathbb{R}$

g. If $U, S \subseteq \mathbf{R}$, U is open, and $U \cap S = \emptyset$, then $U \cap S' = \emptyset$.

True