MATH 2510 **Exam 3** **May 5, 2009** Name Key

You are not to use a calculator, book, notes, or others to do this exam. **Show all essential work.**

1. For each of the following sets, find the supremum, infimum, the boundary points, and the accumulation points. *(10 points)*

   a. $S = (0, 1]$

   $\sup(S) = 1$ $\inf(S) = 0$ $\text{bd}(S) = \{0, 1\}$ $S' = [0, 1]$

   b. $S = \{x \in \mathbf{Q} | x^2 < 5\}$

   $\sup(S) = \sqrt{5}$ $\inf(S) = -\sqrt{5}$ $\text{bd}(S) = [-\sqrt{5}, \sqrt{5}]$ $S' = [-\sqrt{5}, \sqrt{5}]$

2. State the completion axiom. See page 121 *(5 points)*

3. Let $S \subseteq \mathbf{R}$. State the definition of the following. (Don't give an equivalent condition, give the definition.) *(10 points)*

   a. S is open.

   See page 131

   b. S is closed

4. Find the interior of the set $S = [2, 3] \cup [4, 5)$. *(6 points)*

$(2,3) \cup (4,5)$

5. Find all the isolated points in the set $S = \{\frac{3}{n} | n \in \mathbf{N}\} \cup [3, 4]$. *(6 points)*

$S \sim \{3\} = \{\frac{3}{n} \mid n \in \mathbb{N},\ n > 1\}.$

6. Find the closure of the set $S = (-\infty, -1) \cup \{\frac{1}{2^n} | n \in \mathbf{N}\}$. *(6 points)*

$\cancel{B}$ $(-\infty, 1] \cup \{0\} \cup \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$

7. Prove that if $S, U \subseteq \mathbf{R}$ with U open and $S \cap U = \emptyset$, then $S' \cap U = \emptyset$. *(12 points)*

Let $x \in U$. Since U is open x is an interior point of U.

$\therefore \exists \varepsilon > 0 \ni N(x;\varepsilon) \subseteq U$.

$\therefore N(x;\varepsilon) \subseteq \mathbb{R} - S$ since $S \cap U = \phi$.

$\therefore N(x;\varepsilon) \cap S = \phi$

$\therefore N^*(x;\varepsilon) \cap S = \phi$.

So $x \notin S'$.

8. Prove that if $x \in S' - S$, then $x \in \mathrm{bd}(S)$. *(12 points)*

Let $x \in S' - S$.

Let $\varepsilon > 0$. Since $x \in S'$, $N^*(x;\varepsilon) \cap S \neq \phi$.

$N^*(x;\varepsilon) \cap S \subseteq N(x;\varepsilon) \cap S)$. $\therefore N(x;\varepsilon) \cap S \neq \phi$.

Since $x \notin S$, $x \in \mathbb{R} - S$. $\therefore x \in N(x;\varepsilon) \cap (\mathbb{R} - S)$.

$\therefore N(x;\varepsilon) \cap (\mathbb{R} - S) \neq \phi$.

Since $N(x;\varepsilon) \cap S \neq \phi$ and $N(x;\varepsilon) \cap (\mathbb{R} - S) \neq \phi$ $\forall \varepsilon > 0$,

$x \in \mathrm{bd}(S)$.

9. Prove that if $\sup(S)$ exists and $T = \{x \in \mathbf{R} | -x \in S\}$, then $\inf(T)$ exists and $\inf(T) = -\sup(S)$. *(12 points)*

We must show $-\sup(S)$ is a lower bound for T and no other lower bound for T is greater than $-\sup(S)$.

1) We show $-\sup(S)$ is a lower bound for T.
Let $x \in T$. Then $-x \in S$. $\therefore -x \leq \sup(S)$ $\therefore x \geq -\sup(S)$.
$\therefore -\sup(S)$ is a lower bound for ~~S~~ T.

2) We show that if $m > -\sup(S)$, then m is not a lower bound for T.
Let $m > -\sup(S)$ $\therefore -m < \sup(S)$. $\therefore \exists\, y \in S \ni -m < y \leq \sup(S)$.
$\therefore m > -y \in T$.
$\therefore m$ is not a lower bound for T. $\therefore -\sup(S)$ is the greatest lower bound for T.
$\therefore \inf(T) = -\sup(S)$.

10. True or False? If true, then just state that it is true. If false, then state it is false and either change the statement in some nontrivial way to make it true or else give a counter example. *(24 points)*

a. For any set $S \subseteq \mathbf{R}$, $\mathrm{bd}(S) = S'$.

False
$S = \{0\}$. $\mathrm{bd}(S) = \{0\}$, $S' = \emptyset$.

b. There is a subset of $\mathbf{R}$ which is both open and closed.

True
$\mathbb{R}$

c. For any set $S \subseteq \mathbf{R}$, S is open or S is closed.

False
$S = [0, 1)$.

d. For any nonempty set $S \subseteq \mathbf{R}$, if S has both a maximum and a minimum, then S is closed.

False
$S = [0,1) \cup (2,3]$.

e. For any subset $S \subseteq \mathbf{R}$, $\mathrm{bd}(S) = \mathrm{bd}(\mathbf{R} - S)$.

True

f. For any subset $S \subseteq \mathbf{R}$, $S' = (\mathbf{R} - S)'$.

False
$S = \{0\}$ $\quad S' = \emptyset$
$(\mathbb{R} - S)' = \mathbb{R}$