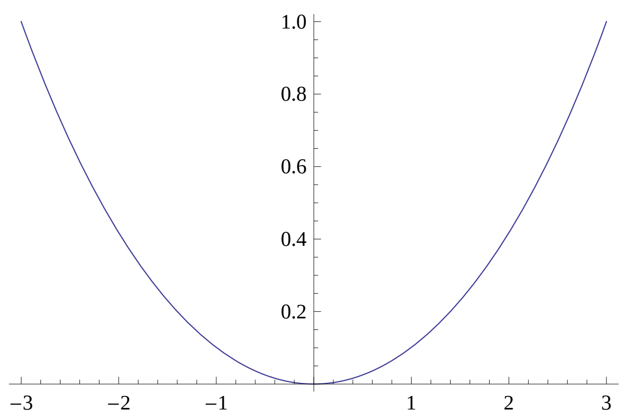
Math 1720 Project - The Perfect Bowl

The Story. John is a wood turner who makes nice bowls, but one day he wondered if he could improve the form (shape) of his bowls. So he read a few books and learned that a lot of wood turners make their bowls in the shape of the bottom of a sphere or else as part of the hyperbolic cosine function (also known as a catenary curve) rotated about the $y-$axis[1]. John has a source of wood that is two inches thick that he decided to use to make bowls modeled on both of these curves. To make a spherical shape he constructed a template in the shape of a circle of the right diameter and to make a catenary curve, he suspended a chain in front of a paper and traced the chain's position.

After making several curves to fit different bowl diameters, John made an interesting discovery. He noticed that as long as the diameter was sufficiently large, the difference between the circle and the catenary was smaller than the width of the pencil mark. He was very curious why this happens, but he doesn't remember much about second semester calculus, so he decided to go to you for help in analyzing the problem.

The problem. Given a bowl, the aspect ratio is the diameter divided by the height. For the shape in the graph below the aspect ratio would be $\frac{6}{1} = 6$. For convenience, you can assume the height of a bowl is 1 unit and then vary the aspect ratio by simply varying the radius. The problem you are asked to solve is to find for which any aspect ratio larger than that number the graphs of the circle and the catenary (each passing through the points $(0,0), (a,1), (-a,1)$) are very close together. By very close together, we will (arbitrarily) mean the y values on the two curves differ by no more than 0.01 for $0 \leq x \leq a$. Also, make a chart showing for various values of a the maximum difference between the $y-$values of the two curves for $0 \leq x \leq a$ and prove that the maximum difference of $y-$values approaches 0 as a goes to infinity.



Suggested Procedure. There are many ways to approach the problem and you can decide which way you wish to proceed. However, you are to do a careful mathematical analysis and prove all the statements you make. For example, it is not sufficient to graph curves and when you see that the curves are within 0.05 of each other just use that aspect ratio because you can't be sure that if you increase the aspect ratio, they may get further apart. Below you will find an outline of one solution which you may wish to follow.

[1] R1, page 48

At some point or points along the way, it may be helpful to use some standard mathematical facts. They include the following:

1. The triangle inequality says that if a and b are any numbers, then $|a+b| \leq |a| + |b|$. This is a standard fact that you need not prove.
2. For any value of $x > 0$, $\ln(1+x) < x$. If you use this fact, be sure to prove it.
3. The equation of a catenary is

$$c(x) = \frac{b}{2}\left(e^{\frac{x}{b}} + e^{\frac{-x}{b}} - 2\right).$$

This is the definition of a catenary, so there is nothing to prove here. The number b is simply a scaling of the graph.

Here are some steps that could lead to a solution.

1. Assuming a catenary bowl has height 1, then the radius of the bowl a is the positive value of x where $c(x) = 1$. Find a relation between a and b and solve for a so that a is a function of b.
2. Find the equation of the circle through $(0,0), (a,1), (-a,1)$ and find the function $s(x)$ that gives the bottom half of the circle. Your answer will involve the number a. What is the relation between the number a and the radius of the circle r?
3. Now write the MacLaurin series for each function $s(x)$ and $c(x)$. Let $p_s(x)$ be the quadratic part of the MacLaurin series for the circle and $p_c(x)$ be the quadratic part of the MacLaurin series for $c(x)$. Then the remainder terms $R_s(x) = s(x) - p_s(x)$ and $R_c(x) = c(x) - p_c(x)$ can be thought of as the sum of the rest of the terms in the series or as the error term in the remainder theorem.
4. Show that the signs of all the coefficients of the series for R_s and R_c are positive. Explain why that shows that both the remainder term functions are increasing for $x > 0$. Show that the function $|p_s(x) - p_c(x)|$ is also increasing for $x > 0$.
5. To estimate $|s(x) - c(x)|$ for $0 \leq x \leq a$. Try the following:
   a. Show that for any $0 \leq x \leq a$, $|p_s(x) - p_c(x)| \leq |s(x) - c(x)| + |R_s(x) - R_c(x)|$ and conclude that $|p_s(x) - p_c(x)| \leq |R_s(a) - R_c(a)|$
   b. Start with $|s(x) - c(x)| = |(p_s(x) + R_s(x)) - (p_c(x) + R_c(x))|$ and do some work to show that $|s(x) - c(x) \leq 2\max|R_s(x) - R_c(x)|$ where the maximum is taken over all $0 \leq x \leq a$.
6. Use what you learned to estimate a number M so that for $a \geq M$, you are sure that $|s(x) - c(x)| < .01$.

References

H. Hayes, Derek, **Woodturning Design**, The Taunton Press, Newtown Connecticut, 2011.

R1. Raffan, Richard, **The Art of Turned Bowls, Designing Bowls with a World-Class Turner**, The Tauton Press, Newtown,Connecticut, 2008.

R2. Raffan, Richard, **Turned-Bowl Design**, The Tauton Press, Newtown, Connecticut, 1987.