

Prove the following statements **using the method of mathematical induction**.

1. Guess the formula for  $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n(n+1)}$  by looking at small values of  $n$  and then prove your answer is correct for any positive value of  $n$  using induction.
2. For every positive integer  $n$ ,  $\sum_{k=1}^n \frac{1}{2^k} = 1 - \frac{1}{2^n}$ .
3.  $1 \times 1! + 2 \times 2! + \cdots + n \times n! = (n+1)! - 1$  for any natural number  $n$ .
4.  $\sum_{k=0}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2$  for any positive integer  $n$ .
5. For any natural number  $n$ ,  $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$ .
6. For any natural number  $n$ ,  $2 + 9 + 16 + \cdots + (7n-5) = \frac{n(7n-3)}{2}$ .
7. The number  $n^3 - n$  is divisible by three for any positive integer  $n$ .
8. The number  $n^5 + 4n$  is divisible by 5 for any natural number  $n$ .
9. The sum of the angles of any polygon with  $n$  sides is  $\pi(n-2)$ .
10. Any postage of over 7 cents can be paid using only 3 cent stamps and 5 cent stamps.
11. For every positive integer  $n$ ,  $n^2 \geq n$ .
12. For any natural number  $n$ ,  $3n \leq 3^n$ .
13. For any real number  $x \geq -1$  and any natural number  $n$ ,  $(1+x)^n \geq 1+nx$ .
14. For any natural number  $n$  and any real number  $x$ ,  $|\sin nx| \leq n|\sin x|$ .
15. For any natural number  $n$ ,  $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$ .
16. Define the Fibonacci sequence by  $f_0 = 1$ ,  $f_1 = 1$ , and for  $n \geq 2$ ,  $f_n = f_{n-1} + f_{n-2}$ . Prove for any natural number  $n$ ,  $f_0 + f_1 + f_2 + f_3 + \cdots + f_n = f_{n+2} - 1$ .
17. Using the Fibonacci sequence defined in the previous problem, prove that  $\sum_{k=0}^n f_k^2 = f_n f_{n+1}$  for any natural number  $n$ .
18. Using the notation in the previous two problems show that  $f_n = \frac{1}{\sqrt{5}} \left[ \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right]$  for any natural number  $n$ .