Deformations of Skew Group Algebras

Briana Foster-Greenwood

November 19, 2010

Skew group algebras

group G	$\xrightarrow{\text{acts on}}$	vectorspace V
group algebra $\mathbb{C}G$	$\xrightarrow{\text{acts on}}$	polynomial algebra $S(V) \cong \mathbb{C}[v_1, \ldots, v_n]$

Skew group algebra $S(V)\#G$

Elements: $\mathbb{C}$-linear combos of monomials $v_1^{e_1} \cdots v_n^{e_n}\mathbf{g}$

Relations:

- $vw - wv = 0$
- group elements multiply as in group
- $\mathbf{g}v = \vec{g}(v)\mathbf{g}$

Example skew group algebra

group $G = \{\mathbf{1}, \mathbf{(123)}, \mathbf{(321)}\}$

group algebra
$\mathbb{C}G$

acts by
permuting
basis vectors
$\longrightarrow$

vectorspace $V \cong \mathbb{C}^3$

polynomial algebra
$S(V) \cong \mathbb{C}[v_1, v_2, v_3]$

skew group algebra
$S(V)\#G$

sample element:

$$(v_1 v_2^2 + 5v_3)\mathbf{1} + (v_1 v_2 v_3)\mathbf{(123)} + (v_1^3 - v_2)\mathbf{(321)}$$

Example skew group algebra

group $G = \{\mathbf{1}, \mathbf{(123)}, \mathbf{(321)}\}$

group algebra
$\mathbb{C}G$

acts by
permuting
basis vectors
$\longrightarrow$

vectorspace $V \cong \mathbb{C}^3$

polynomial algebra
$S(V) \cong \mathbb{C}[v_1, v_2, v_3]$

skew group algebra
$S(V)\#G$

sample multiplication:

$$\mathbf{(123)}(v_1 v_2 v_3^5)\mathbf{(321)} = (v_2 v_3 v_1^5)\mathbf{(123)(321)} = (v_2 v_3 v_1^5)\mathbf{1} = (v_1^5 v_2 v_3)\mathbf{1}$$

PBW property

$S(V)\#G$ satisfies PBW property, i.e. monomials

$$v_1^{e_1} \cdots v_n^{e_n} \mathbf{g}$$

form a vectorspace basis of $S(V)\#G$

Deforming the commutator relation

- Let κ be a map from $V \times V$ to the group algebra $\mathbb{C}G$.
- Try to use map κ to get a deformation of $S(V)\#G$.

$S(V)\#G$

Relations:

- $vw - wv = 0$
- group elements multiply as in group
- $\mathbf{g}v = \vec{g}(v)\mathbf{g}$

$(S(V)\#G)_\kappa$

Relations:

- $vw - wv = \kappa(v, w)$
- group elements multiply as in group
- $\mathbf{g}v = \vec{g}(v)\mathbf{g}$

Is $(S(V)\#G)_\kappa$ a deformation of $S(V)\#G$?

Results of Ram/Shepler, Witherspoon, and others

$(S(V)\#G)_\kappa$ is a deformation of $S(V)\#G$

$$\Updownarrow$$

$(S(V)\#G)_\kappa$ satisfies PBW property, i.e. monomials

$$v_1^{e_1} \cdots v_n^{e_n} \mathbf{g}$$

form a vectorspace basis of $(S(V)\#G)_\kappa$

$$\Updownarrow$$

κ is skew-symmetric bilinear and compatible with group action

Properties of $\kappa : V \times V \to \mathbb{C}G$

κ is skew-symmetric bilinear

$$vw - wv = \kappa(v, w)$$

- Skew-symmetry: swapping v and w introduces a minus sign, so

$$\kappa(v, w) = -\kappa(w, v)$$

- Bilinearity: commutator is bilinear, so κ must be bilinear

e.g. can compute $\kappa(3v_1 - 5v_2, 2w_1 + w_2)$ by freshman FOIL:

$$\underbrace{6\kappa(v_1, w_1)}_{F} + \underbrace{3\kappa(v_1, w_2)}_{O} - \underbrace{10\kappa(v_2, w_1)}_{I} - \underbrace{5\kappa(v_2, w_2)}_{L}$$

Properties of $\kappa : V \times V \to \mathbb{C}G$

Components κ_g are skew-symmetric bilinear

- Break κ into $\mathbf{g}$-components:

$$\kappa(v, w) = \sum_{g \in G} \kappa_g(v, w)\mathbf{g}$$

- Each κ_g is a skew-symmetric bilinear map from $V \times V$ to $\mathbb{C}$

Properties of $\kappa : V \times V \to \mathbb{C}G$

Which κ maps are compatible with group action?

Condition 1 (invariant theory)

For all group elements g and h and all vectors v and w:

$$\kappa_g(v, w) = \kappa_{hgh^{-1}}\left(\vec{h}(v), \vec{h}(w)\right)$$

This implies κ_1 is G-invariant.

Properties of $\kappa : V \times V \to \mathbb{C}G$

Which κ maps are compatible with group action?

Condition 2 (linear algebra)

For all group elements g and vectors v_1, v_2, v_3:

$$\begin{aligned} & \kappa_g(v_2, v_3)(\vec{g}(v_1) - v_1) \\ + \quad & \kappa_g(v_3, v_1)(\vec{g}(v_2) - v_2) \\ + \quad & \kappa_g(v_1, v_2)(\vec{g}(v_3) - v_3) = 0 \end{aligned}$$

This is a linear relation among vectors. If $g = 1$, this condition says $0 = 0$.

Example deformations of $S(V)\#G$

G acts on V by left regular rep

Example (Setup for $S(V)\#G$)

- group: (almost) any finite group G
- vectorspace: V with basis $\{v_g \ : \ g \in G\}$
- action: left regular action, $\vec{g}(v_x) = v_{gx}$

For G acting on V by regular representation:

$(S(V)\#G)_\kappa$ is a deformation of $S(V)\#G$ if and only if...

- κ_1 is G-invariant, skew-symmetric, bilinear
- $\kappa_g = 0$ for $g \neq 1$

$\kappa_g = 0$ for $g \neq 1$

G acts on V by left regular rep

Example ($|G| = 8$ and g order 4)

- work with new basis: use eigenvectors of g
- g has eigenvalues $1, 1, i, i, -1, -1, -i, -i$
- pick any two eigenvectors v_1 and v_2 from basis
- can find third w in basis with $\vec{g}(w) = \lambda w$ and $\lambda \neq 1$
- linear algebra condition reads

$$(\text{scalar})v_1 + (\text{scalar})v_2 + \kappa_g(v_1, v_2)\underbrace{(\lambda - 1)}_{\text{nonzero}} w = 0$$

- linear independence implies $\kappa_g(v_1, v_2) = 0$

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1								
v_{-1}								
v_i								
v_{-i}								
v_j								
v_{-j}								
v_k								
v_{-k}								

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

-
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0							
v_{-1}		0						
v_i			0					
v_{-i}				0				
v_j					0			
v_{-j}						0		
v_k							0	
v_{-k}								0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

-
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0							
v_{-1}		0						
v_i			0					
v_{-i}				0				
v_j					0			
v_{-j}						0		
v_k							0	
v_{-k}								0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i)$
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a					
v_{-1}		0						
v_i			0					
v_{-i}				0				
v_j					0			
v_{-j}						0		
v_k							0	
v_{-k}								0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i)$
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a					
v_{-1}		0						
v_i			0					
v_{-i}				0				
v_j					0			
v_{-j}						0		
v_k							0	
v_{-k}								0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i})$
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a					
v_{-1}		0		a				
v_i			0					
v_{-i}				0				
v_j					0			
v_{-j}						0		
v_k							0	
v_{-k}								0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i})$
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a					
v_{-1}		0		a				
v_i			0					
v_{-i}				0				
v_j					0			
v_{-j}						0		
v_k							0	
v_{-k}								0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1})$
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a					
v_{-1}		0		a				
v_i		a	0					
v_{-i}				0				
v_j					0			
v_{-j}						0		
v_k							0	
v_{-k}								0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1})$
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a					
v_{-1}		0		a				
v_i		a	0					
v_{-i}				0				
v_j					0			
v_{-j}						0		
v_k							0	
v_{-k}								0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1)$
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a					
v_{-1}		0		a				
v_i		a	0					
v_{-i}	a			0				
v_j					0			
v_{-j}						0		
v_k							0	
v_{-k}								0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1)$
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a					
v_{-1}		0		a				
v_i		a	0					
v_{-i}	a			0				
v_j					0			
v_{-j}						0		
v_k							0	
v_{-k}								0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a					
v_{-1}		0		a				
v_i		a	0					
v_{-i}	a			0				
v_j					0			a
v_{-j}						0	a	
v_k					a		0	
v_{-k}						a		0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a	$-a$				
v_{-1}		0	$-a$	a				
v_i	$-a$	a	0					
v_{-i}	a	$-a$		0				
v_j					0		$-a$	a
v_{-j}						0	a	$-a$
v_k					a	$-a$	0	
v_{-k}					$-a$	a		0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
-
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a	$-a$				
v_{-1}		0	$-a$	a				
v_i	$-a$	a	0					
v_{-i}	a	$-a$		0				
v_j					0		$-a$	a
v_{-j}						0	a	$-a$
v_k					a	$-a$	0	
v_{-k}					$-a$	a		0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
- $\kappa_1(v_1, v_j)$
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a	$-a$	b			
v_{-1}		0	$-a$	a				
v_i	$-a$	a	0					
v_{-i}	a	$-a$		0				
v_j					0		$-a$	a
v_{-j}						0	a	$-a$
v_k					a	$-a$	0	
v_{-k}					$-a$	a		0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
- $\kappa_1(v_1, v_j)$
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a	$-a$	b			
v_{-1}		0	$-a$	a				
v_i	$-a$	a	0					
v_{-i}	a	$-a$		0				
v_j					0		$-a$	a
v_{-j}						0	a	$-a$
v_k					a	$-a$	0	
v_{-k}					$-a$	a		0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
- $\kappa_1(v_1, v_j) = \kappa_1(v_{-1}, v_{-j})$
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a	$-a$	b			
v_{-1}		0	$-a$	a		b		
v_i	$-a$	a	0					
v_{-i}	a	$-a$		0				
v_j					0		$-a$	a
v_{-j}						0	a	$-a$
v_k					a	$-a$	0	
v_{-k}					$-a$	a		0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
- $\kappa_1(v_1, v_j) = \kappa_1(v_{-1}, v_{-j})$
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a	$-a$	b			
v_{-1}		0	$-a$	a		b		
v_i	$-a$	a	0					
v_{-i}	a	$-a$		0				
v_j					0		$-a$	a
v_{-j}						0	a	$-a$
v_k					a	$-a$	0	
v_{-k}					$-a$	a		0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
- $\kappa_1(v_1, v_j) = \kappa_1(v_{-1}, v_{-j}) = \kappa_1(v_i, v_k) = \kappa_1(v_{-i}, v_{-k}) = \kappa_1(v_j, v_{-1}) = \cdots$
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a	$-a$	b			
v_{-1}		0	$-a$	a		b		
v_i	$-a$	a	0				b	
v_{-i}	a	$-a$		0				b
v_j		b			0		$-a$	a
v_{-j}	b					0	a	$-a$
v_k				b	a	$-a$	0	
v_{-k}			b		$-a$	a		0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
- $\kappa_1(v_1, v_j) = \kappa_1(v_{-1}, v_{-j}) = \kappa_1(v_i, v_k) = \kappa_1(v_{-i}, v_{-k}) = \kappa_1(v_j, v_{-1}) = \cdots$
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a	$-a$	b	$-b$		
v_{-1}		0	$-a$	a	$-b$	b		
v_i	$-a$	a	0				b	$-b$
v_{-i}	a	$-a$		0			$-b$	b
v_j	$-b$	b			0		$-a$	a
v_{-j}	b	$-b$				0	a	$-a$
v_k			$-b$	b	a	$-a$	0	
v_{-k}			b	$-b$	$-a$	a		0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
- $\kappa_1(v_1, v_j) = \kappa_1(v_{-1}, v_{-j}) = \kappa_1(v_i, v_k) = \kappa_1(v_{-i}, v_{-k}) = \kappa_1(v_j, v_{-1}) = \cdots$
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a	$-a$	b	$-b$	c	
v_{-1}		0	$-a$	a	$-b$	b		c
v_i	$-a$	a	0			c	b	$-b$
v_{-i}	a	$-a$		0	c		$-b$	b
v_j	$-b$	b	c		0		$-a$	a
v_{-j}	b	$-b$		c		0	a	$-a$
v_k		c	$-b$	b	a	$-a$	0	
v_{-k}	c		b	$-b$	$-a$	a		0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
- $\kappa_1(v_1, v_j) = \kappa_1(v_{-1}, v_{-j}) = \kappa_1(v_i, v_k) = \kappa_1(v_{-i}, v_{-k}) = \kappa_1(v_j, v_{-1}) = \cdots$
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a	$-a$	b	$-b$	c	$-c$
v_{-1}		0	$-a$	a	$-b$	b	$-c$	c
v_i	$-a$	a	0		$-c$	c	b	$-b$
v_{-i}	a	$-a$		0	c	$-c$	$-b$	b
v_j	$-b$	b	c	$-c$	0		$-a$	a
v_{-j}	b	$-b$	$-c$	c		0	a	$-a$
v_k	$-c$	c	$-b$	b	a	$-a$	0	
v_{-k}	c	$-c$	b	$-b$	$-a$	a		0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
- $\kappa_1(v_1, v_j) = \kappa_1(v_{-1}, v_{-j}) = \kappa_1(v_i, v_k) = \kappa_1(v_{-i}, v_{-k}) = \kappa_1(v_j, v_{-1}) = \cdots$
-

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0		a	$-a$	b	$-b$	c	$-c$
v_{-1}		0	$-a$	a	$-b$	b	$-c$	c
v_i	$-a$	a	0		$-c$	c	b	$-b$
v_{-i}	a	$-a$		0	c	$-c$	$-b$	b
v_j	$-b$	b	c	$-c$	0		$-a$	a
v_{-j}	b	$-b$	$-c$	c		0	a	$-a$
v_k	$-c$	c	$-b$	b	a	$-a$	0	
v_{-k}	c	$-c$	b	$-b$	$-a$	a		0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
- $\kappa_1(v_1, v_j) = \kappa_1(v_{-1}, v_{-j}) = \kappa_1(v_i, v_k) = \kappa_1(v_{-i}, v_{-k}) = \kappa_1(v_j, v_{-1}) = \cdots$
- $\kappa_1(v_1, v_{-1}) = \kappa_1(v_{-1}, v_1) = -\kappa_1(v_{-1}, v_1)$

Finding a G-invariant map κ_1

Quaternion group acts on $V \cong \mathbb{C}^8$ by left regular rep

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

Record $\kappa_1(v_x, v_y)$ for pairs of basis vectors:

	v_1	v_{-1}	v_i	v_{-i}	v_j	v_{-j}	v_k	v_{-k}
v_1	0	0	a	$-a$	b	$-b$	c	$-c$
v_{-1}	0	0	$-a$	a	$-b$	b	$-c$	c
v_i	$-a$	a	0	0	$-c$	c	b	$-b$
v_{-i}	a	$-a$	0	0	c	$-c$	$-b$	b
v_j	$-b$	b	c	$-c$	0	0	$-a$	a
v_{-j}	b	$-b$	$-c$	c	0	0	a	$-a$
v_k	$-c$	c	$-b$	b	a	$-a$	0	0
v_{-k}	c	$-c$	b	$-b$	$-a$	a	0	0

G-invariance: $\kappa_1(v_x, v_y) = \kappa_1(v_{hx}, v_{hy})$

- $\kappa_1(v_1, v_i) = \kappa_1(v_{-1}, v_{-i}) = \kappa_1(v_i, v_{-1}) = \kappa_1(v_{-i}, v_1) = \kappa_1(v_j, v_{-k}) = \cdots$
- $\kappa_1(v_1, v_j) = \kappa_1(v_{-1}, v_{-j}) = \kappa_1(v_i, v_k) = \kappa_1(v_{-i}, v_{-k}) = \kappa_1(v_j, v_{-1}) = \cdots$
- $\kappa_1(v_1, v_{-1}) = \kappa_1(v_{-1}, v_1) = -\kappa_1(v_{-1}, v_1)$

κ_1 and the right regular rep

Quaternions act on $V \cong \mathbb{C}^8$ by left regular rep

Let $[g]$ be matrix for the right regular action $\vec{g}(v_x) = v_{xg^{-1}}$. For example:

$$[\,i\,] = \begin{pmatrix} & & 1 & 0 & & & & \\ & & 0 & 1 & & & & \\ 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \\ & & & & 1 & 0 & & \\ & & & & 0 & 1 & & \end{pmatrix}$$

κ_1 and the right regular rep

Quaternions act on $V \cong \mathbb{C}^8$ by left regular rep

Let $[g]$ be matrix for the right regular action $\vec{g}(v_x) = v_{xg^{-1}}$. For example:

$$[\,i\,] = \begin{pmatrix} & & 1 & 0 & & & & \\ & & 0 & 1 & & & & \\ 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \\ & & & & 1 & 0 & & \\ & & & & 0 & 1 & & \end{pmatrix}$$

Compare with κ_1...

$$\kappa_1 = \begin{pmatrix} 0 & 0 & a & -a & b & -b & c & -c \\ 0 & 0 & -a & a & -b & b & -c & c \\ -a & a & 0 & 0 & c & -c & b & -b \\ a & -a & 0 & 0 & -c & c & -b & b \\ -b & b & c & -c & 0 & 0 & -a & a \\ b & -b & -c & c & 0 & 0 & a & -a \\ -c & c & -b & b & a & -a & 0 & 0 \\ c & -c & b & -b & -a & a & 0 & 0 \end{pmatrix}$$

κ_1 and the right regular rep

Quaternions act on $V \cong \mathbb{C}^8$ by left regular rep

Let $[g]$ be matrix for the right regular action $\vec{g}(v_x) = v_{xg^{-1}}$. For example:

$$[\,i\,] = \begin{pmatrix} & & 1 & 0 & & & & \\ & & 0 & 1 & & & & \\ 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \\ & & & & 1 & 0 & & \\ & & & & 0 & 1 & & \end{pmatrix}$$

Compare with κ_1...

$$\kappa_1 = \begin{pmatrix} 0 & 0 & a & -a & b & -b & c & -c \\ 0 & 0 & -a & a & -b & b & -c & c \\ -a & a & 0 & 0 & c & -c & b & -b \\ a & -a & 0 & 0 & -c & c & -b & b \\ -b & b & c & -c & 0 & 0 & -a & a \\ b & -b & -c & c & 0 & 0 & a & -a \\ -c & c & -b & b & a & -a & 0 & 0 \\ c & -c & b & -b & -a & a & 0 & 0 \end{pmatrix}$$

κ_1 is a linear combo of skew-symmetric matrices

$$\kappa_1 = a([i] - [i]^T) + b([j] - [j]^T) + c([k] - [k]^T)$$

Describing deformations $(S(V)\#G)_\kappa$ of $S(V)\#G$

G acts on V by left regular rep

Proposition

If G acts on V via the left regular representation, then the κ_1's yielding a deformation $(S(V)\#G)_\kappa$ of $S(V)\#G$ are the skew-symmetric matrices

$$\kappa_1 = \sum_{g \in G} c_g([g] - [g]^T),$$

where the c_g's are complex scalars, and $[g]$ is the matrix of g acting on V via the right regular representation.

Describing deformations $(S(V)\#G)_\kappa$ of $S(V)\#G$

G acts on V by left regular rep

Proposition

If G acts on V via the left regular representation, then the κ_1's yielding a deformation $(S(V)\#G)_\kappa$ of $S(V)\#G$ are the skew-symmetric matrices

$$\kappa_1 = \sum_{g \in G} c_g([g] - [g]^T),$$

where the c_g's are complex scalars, and $[g]$ is the matrix of g acting on V via the right regular representation.

Observations:

Describing deformations $(S(V)\#G)_\kappa$ of $S(V)\#G$

G acts on V by left regular rep

Proposition

If G acts on V via the left regular representation, then the κ_1's yielding a deformation $(S(V)\#G)_\kappa$ of $S(V)\#G$ are the skew-symmetric matrices

$$\kappa_1 = \sum_{g \in G} c_g([g] - [g]^T),$$

where the c_g's are complex scalars, and $[g]$ is the matrix of g acting on V via the right regular representation.

Observations:

- $[g]$ is a permutation matrix, $[g^{-1}] = [g]^T$

Describing deformations $(S(V)\#G)_\kappa$ of $S(V)\#G$

G acts on V by left regular rep

Proposition

If G acts on V via the left regular representation, then the κ_1's yielding a deformation $(S(V)\#G)_\kappa$ of $S(V)\#G$ are the skew-symmetric matrices

$$\kappa_1 = \sum_{g \in G} c_g([g] - [g]^T),$$

where the c_g's are complex scalars, and $[g]$ is the matrix of g acting on V via the right regular representation.

Observations:

- $[g]$ is a permutation matrix, $[g^{-1}] = [g]^T$
- if $g^2 = 1$, then $[g] - [g]^T = 0$

Describing deformations $(S(V)\#G)_\kappa$ of $S(V)\#G$

G acts on V by left regular rep

Proposition

If G acts on V via the left regular representation, then the κ_1's yielding a deformation $(S(V)\#G)_\kappa$ of $S(V)\#G$ are the skew-symmetric matrices

$$\kappa_1 = \sum_{g \in G} c_g([g] - [g]^T),$$

where the c_g's are complex scalars, and $[g]$ is the matrix of g acting on V via the right regular representation.

Observations:

- $[g]$ is a permutation matrix, $[g^{-1}] = [g]^T$
- if $g^2 = 1$, then $[g] - [g]^T = 0$
- if $g^2 \neq 1$, terms $c_g([g] - [g]^T)$ and $c_{g^{-1}}([g]^T - [g])$ combine

How many deformations of type $(S(V)\#G)_\kappa$?

G acts on V by left regular rep

Corollary

If G acts on V by the left regular representation, then the $\mathbb{C}$-vectorspace of κ's such that $(S(V)\#G)_\kappa$ is a deformation of $S(V)\#G$ has dimension

$$\frac{|G| - \#\{g \in G \,:\, g^2 = 1\}}{2}$$

Example ($G = \{\pm 1, \pm i, \pm j, \pm k\}$)

$$\frac{8-2}{2} = 3$$

Hochschild cohomology point of view

G acts on V by left regular rep

- theory of Hochschild cohomology for skew group algebras predicts

vectorspace of κ's such that $(S(V)\#G)_\kappa$ is a deformation of $S(V)\#G$	$\longleftrightarrow$	$\left(\bigwedge^2 V^*\right)^G$

- character theory computes $\dim_{\mathbb{C}} \left(\bigwedge^2 V^*\right)^G$

Thanks!