

Invariant Theory and Hochschild Cohomology of Skew Group Algebras

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Outline

- 1 Hochschild Cohomology and Associative Deformations
- 2 Hochschild Cohomology of Skew Group Algebras
- 3 Classical invariant theory tools

Hochschild Cohomology and Associative Deformations

Deformations of algebras

Start with an associative \mathbb{F} -algebra: A

Adjoin a central parameter: $A[t]$
(this will be underlying vector space structure)

Define a new multiplication: $a * b = ab + \mu_1(a, b)t + \mu_2(a, b)t^2 + \dots$
(first define for pairs of elements in A ; then extend $\mathbb{F}[t]$ -bilinearly)

Specialize to $t \in \mathbb{F}$ to get many algebras

$t = 0 \rightarrow$ original algebra

$t = 1 \rightarrow ???$

Hochschild cohomology and associative deformations

$A \overset{\text{deform}}{\rightsquigarrow} A[t]$ with multiplication $a * b = ab + \mu_1(a, b)t + \mu_2(a, b)t^2 + \dots$

- picking any old μ_k 's will not usually yield an **associative** algebra
- to even get started, μ_1 must be a **Hochschild 2-cocycle**:

$$a\mu_1(b, c) - \mu_1(ab, c) + \mu_1(a, bc) - \mu_1(a, b)c = 0$$

What is Hochschild cohomology?

Hochschild cohomology of an algebra A

- 1-maps: $A \rightarrow A$
- 2-maps: $A \otimes A \rightarrow A$
- 3-maps: $A \otimes A \otimes A \rightarrow A$
- \vdots
- $\mathrm{HH}^k(A)$ = equivalence classes of k -maps satisfying cocycle conditions

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$$\mathrm{HH}^0(A) = \text{center of } A$$

$$\mathrm{HH}^1(A) = \frac{\text{derivations}}{\text{inner derivations}} \quad (\text{can use to construct some 2-cocycles!})$$

$$\mathrm{HH}^2(A) \supset \text{first multiplication maps of deformations}$$

$$\mathrm{HH}^3(A) \supset \text{"obstructions" to obtaining associative deformations}$$

Hochschild Cohomology of Skew Group Algebras

Skew group algebras

group G

acts on
 \longrightarrow

vectorspace V

group algebra
 $\mathbb{C}G$

acts on
 \longrightarrow

polynomial algebra
 $S(V) \cong \mathbb{C}[v_1, \dots, v_n]$

Skew group algebra $S(V) \# G$

Elements: \mathbb{C} -linear combos of monomials $v_1^{e_1} \cdots v_n^{e_n} \mathbf{g}$

Relations:

- $vw - wv = 0$
- group elements multiply as in group
- $\mathbf{g}v = \vec{g}(v)\mathbf{g}$

$\mathrm{HH}^0(S(V)\#G)$ - Invariant polynomials

The center of a skew group algebra is the set of G -invariant polynomials:

$$S(V)^G = \{f \in S(V) : \vec{g}(f) = f \text{ for all } g \in G\}$$

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Example - Sym_3 acts on $\mathbb{C}[x, y, z]$ by permuting the variables

- elementary symmetric polynomials

$$f_1 = x + y + z$$

$$f_2 = yz + xz + xy$$

$$f_3 = xyz$$

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- a few invariant derivations

$$\begin{aligned}\theta_0 &= \partial_x + \partial_y + \partial_z \\ \theta_1 &= x\partial_x + y\partial_y + z\partial_z \\ \theta_2 &= yz\partial_x + xz\partial_y + xy\partial_z\end{aligned}$$

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- build more by multiplying by invariant polynomials:

$$f_3\theta_2 = xy^2z^2\partial_x + x^2yz^2\partial_y + x^2y^2z\partial_z$$

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Big invariant theory description of $\mathrm{HH}(S(V)\#G)$

Theorem (Ginzburg-Kaledin, Farinati)

$$\mathrm{HH}^k(S(V)\#G) \cong (S(V) \otimes \bigwedge^k V^*)^G \oplus \text{more!}$$

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The remaining summands are spaces of semi-invariants under centralizer subgroups:

$$\left(S(V^g) \otimes \bigwedge^{k-c} (V^g)^* \otimes \mathbb{C}_\chi \right)^{Z(g)}$$

(one summand per conjugacy class)

Classical invariant theory tools

Graded vector spaces and Poincaré series

Start with a graded vector space:

$$S = \bigoplus_{d \geq 0} S_d \quad (\text{with } S_d S_e \subset S_{d+e})$$

The **Poincaré series** for S is a power series with the coefficient of t^d recording the dimension of S_d :

$$P_t(S) = \sum_{d \geq 0} (\dim_{\mathbb{C}} S_d) t^d$$

Poincaré series for $\mathbb{C}[f_1, f_2, f_3] \subset \mathbb{C}[x, y, z]^{\text{Sym}_3}$

$$f_1 = x + y + z,$$

$$f_2 = yz + xz + xy,$$

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Build more polynomials \rightarrow

Note: f_1, f_2, f_3 are algebraically independent,
which forces all these new polynomials to be
linearly independent.

degree	polynomials
0	1
1	f_1
2	f_1^2, f_2
3	$f_1^3, f_1 f_2, f_3$
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$$P_t(\mathbb{C}[f_1, f_2, f_3]) = 1 + t + 2t^2 + 3t^3 + \dots$$

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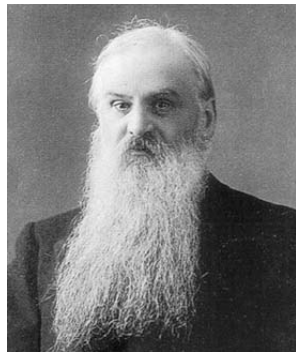
Molien's theorem (1897)

Poincaré series for ring of invariant polynomials

Poincaré series for S^G

$$P_t(S^G) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gt)}$$

$\det(1 - gt)$ is the characteristic polynomial for the matrix of g .



Poincaré series for $\mathbb{C}[x, y, z]^{\text{Sym}_3}$

elements	eigenvalues	characteristic polynomial
1	1, 1, 1	$(1 - t)^3$
(12), (13), (23)	1, 1, -1	$(1 - t)^2(1 + t)$
(123), (321)	1, ω , ω^2	$(1 - t)(1 - \omega t)(1 - \omega^2 t)$

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$$\begin{aligned}
 P_t(S^G) &= \frac{1}{6} \left[\frac{1}{(1 - t)^3} + \frac{3}{(1 - t)^2(1 + t)} + \frac{2}{(1 - t^3)} \right] \\
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By comparing Poincaré series: $\mathbb{C}[f_1, f_2, f_3] = \mathbb{C}[x, y, z]^{\text{Sym}_3}$

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Generalizations of Molien's theorem

Tensoring with an exterior algebra

Poincaré series for $(S(V) \otimes \wedge V^*)^G$

$$P_{X,Y} = \frac{1}{|G|} \sum_{g \in G} \frac{\det(1 + g^* Y)}{\det(1 - gX)}$$

The coefficient of $X^i Y^j$ tells you the dimension of $(S_i(V) \otimes \wedge^j V^*)^G$.

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More generally, the Poincaré series for $(S(V) \otimes \bigwedge V^* \otimes \mathbb{C}_\chi)^G$ is

$$P_{X,Y} = \frac{1}{|G|} \sum_{g \in G} \frac{\chi^*(g) \det(1 + g^* Y)}{\det(1 - gX)}$$

Predictive power of Poincaré

Module structure for identity component of $\mathrm{HH}(S(V)\# \mathrm{Alt}_4)$

Let $G = \mathrm{Alt}_4$ act irreducibly on $V \cong \mathbb{C}^3$.

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invariant polynomials invariant derivations invariant 2-forms invariant 3-forms

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- denominators \rightsquigarrow free module over $R = \mathbb{C}[f_2, f_3, f_4]$

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- numerators \rightsquigarrow how many generators? polynomial degree?

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Invariant derivations:

- the invariant derivations are the free R -span of six derivations of polynomial degrees 1, 2, 3, 3, 4, 5
- it is now a *finite linear algebra problem* to find all invariant derivations

Thanks!