# Graded Hecke Algebras and Reflection Length versus Codimension

## A combinatorial problem motivated by deformation theory

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# Skew group algebras

- finite group $G$ acts linearly on $V \cong \mathbb{C}^n$
- action of $G$ on $V$ extends to action on the symmetric algebra $S(V)$
- the skew group algebra $S(V)\#G$ is the semidirect product algebra $S(V) \rtimes \mathbb{C}G$

# Deformations of skew group algebras



Let $\kappa : V \times V \to \mathbb{C}G$ be skew-symmetric, bilinear.

When $\kappa$ is also *compatible with the group action*, we can define a deformation $(S(V)\#G)_\kappa$ of $S(V)\#G$:

**$S(V)\#G$**

Generators: $v$ in $V$, $\mathbf{g}$ in $G$

Relations:

- $vw - wv = 0$
- group elements multiply as in group
- $\mathbf{g}v = \vec{g}(v)\mathbf{g}$

**$(S(V)\#G)_\kappa$**

Generators: $v$ in $V$, $\mathbf{g}$ in $G$

Relations:

- $vw - wv = \kappa(v, w)$
- group elements multiply as in group
- $\mathbf{g}v = \vec{g}(v)\mathbf{g}$

Deformations of this form are examples of graded Hecke algebras.

# Deformations of skew group algebras

## Example: $G$ acts on $V$ via the left regular rep

Let $G$ act on $V \cong \mathbb{C}^{|G|}$ via the left regular representation.

- theory of Hochschild cohomology for skew group algebras predicts

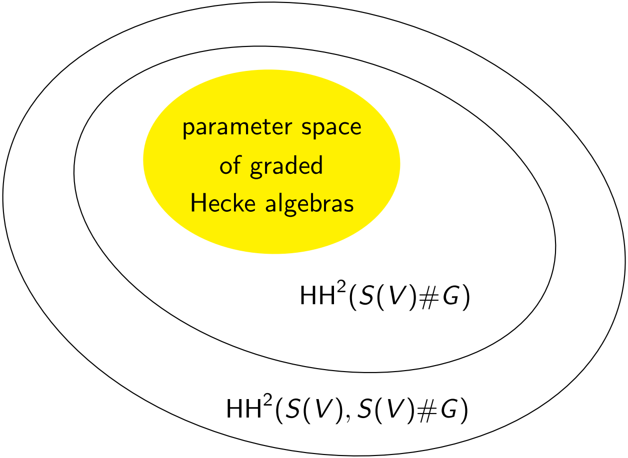
| | | |
|---|---|---|
| vectorspace of $\kappa$'s such that $(S(V)\#G)_\kappa$ is a deformation of $S(V)\#G$ | $=$ | vectorspace of skew-symmetric, $G$-invariant, bilinear forms $\kappa : V \times V \to \mathbb{C}$ |

- character theory computes

$$\dim_{\mathbb{C}}\{\text{GHA's}\} = \frac{|G| - \#\{g \in G \ : \ g^2 = 1\}}{2}$$

# Cohomology detects potential associative deformations

$A$ an algebra, $M$ an $A$-bimodule
Hochschild cohomology: $\mathrm{HH}(A, M) = \mathrm{Ext}_{A \otimes A^{\mathrm{op}}}(A, M)$



# Generating cohomology from atoms in a poset

**Theorem (Shepler-Witherspoon, 2009)**

If $G$ acts faithfully on $V$, then the Hochschild cohomology

$$\mathrm{HH}^\bullet(S(V), S(V)\#G)$$

is generated under cup product by $\mathrm{HH}^\bullet(S(V))$ and volume forms corresponding to the atoms in the codimension poset for $G$.

# Codimension poset

Let $G$ be a group acting on $V \cong \mathbb{C}^n$.

The codimension of an element is the codimension of its fixed point space:

$$\operatorname{codim}(g) = \dim V - \dim\{v \in V \ : \ gv = v\}$$

Define the codimension order on $G$ by:

$$a \leq_{\perp} c \qquad \Leftrightarrow \qquad \operatorname{codim}(a) + \operatorname{codim}(a^{-1}c) = \operatorname{codim}(c)$$

## Example ($G = \mathbb{T}^5$)

Let $\alpha_1$, $\alpha_2$, $\alpha_3$, $\beta_1$, $\beta_2 \neq 1$ be roots of unity.

$$\begin{aligned} a &= \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3, \ 1, \ 1) \\ b &= \operatorname{diag}(\ 1, \ 1, \ 1, \beta_1, \beta_2) \\ c &= \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2) \end{aligned}$$

$a \leq_{\perp} c$ and $b \leq_{\perp} c$

$c$ is not an atom because it has a nontrivial factorization with codimensions adding

# Reflection groups

Let $V \cong \mathbb{C}^n$.

An element of $GL(V)$ is a reflection if it fixes a hyperplane pointwise, i.e., is conjugate to

$$\begin{pmatrix} \zeta & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

for some root of unity $\zeta \neq 1$.

A finite group $G \subset GL(V)$ is a reflection group if it is generated by reflections

*Every reflection (in $G$) is an atom in the codimension poset.*

# Reflection length poset

Let $G$ be a reflection group acting on $V \cong \mathbb{C}^n$.

The (absolute) reflection length of an element $g$ is the minimum number of reflections needed to express $g$ as a product of reflections in $G$:

$$\ell(g) = \min\{k \ : \ g = s_1 \cdots s_k \text{ for some reflections } s_i \text{ in } G\}$$

Define the reflection length order on $G$ by:

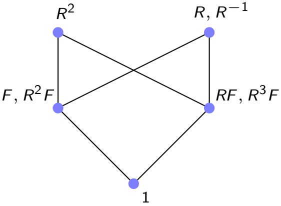
$$a \leq_\ell c \qquad \Leftrightarrow \qquad \ell(a) + \ell(a^{-1}c) = \ell(c)$$

*The only atoms in the reflection length poset are the reflections.*

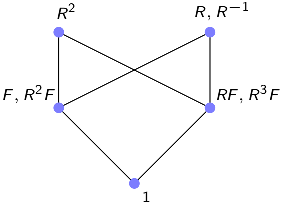
# Reflection length and codimension posets

## Dihedral group of order 8 acts on $V \cong \mathbb{C}^2$

| | $1$ | $R^2$ | $R, R^{-1}$ | $F, R^2F$ | $RF, R^3F$ |
|---|---|---|---|---|---|
| $\ell(g)$ | 0 | 2 | 2 | 1 | 1 |
| $\mathrm{codim}(g)$ | 0 | 2 | 2 | 1 | 1 |




Reflection length poset

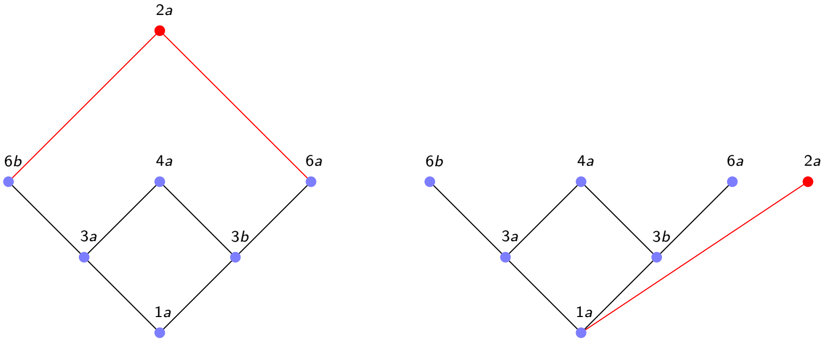



Codimension poset

# Reflection length and codimension posets

## Complex reflection group $G_4$ acts on $V \cong \mathbb{C}^2$

| | $1a$ | $2a$ | $4a$ | $3a$ | $3b$ | $6a$ | $6b$ |
|---|---|---|---|---|---|---|---|
| $\ell(g)$ | 0 | 3 | 2 | 1 | 1 | 2 | 2 |
| codim$(g)$ | 0 | 2 | 2 | 1 | 1 | 2 | 2 |




Reflection length poset

Codimension poset

# Questions

- For which complex reflection groups do the reflection length and codimension functions/posets coincide?
- What are the atoms in the codimension poset?

TFAE:

1. $\ell(g) = \text{codim}(g)$ for every $g$ in $G$.
2. $\ell(g) = \text{codim}(g)$ for every codimension atom $g$ in $G$.
3. Every codimension atom is a reflection.
4. For every $g \neq 1$, there exists a reflection $s$ in $G$ such that $\text{codim}(gs) < \text{codim}(g)$.

# History

Coxeter groups

- Carter (1970's) shows absolute reflection length matches codimension in Weyl groups
- proof works just as well for any Coxeter group

Infinite family $G(r, p, n)$

- Shi (2007) gives formula for reflection length in terms of a maximum over certain partitions
- partition criteria for reflection length to equal codimension
- reflection length coincides with codimension in $G(r, 1, n) \cong (\mathbb{Z}/r\mathbb{Z})^n \rtimes \mathrm{Sym}_n$

Exceptional complex reflection groups $G_4 - G_{37}$ ???

# Infinite family $G(r,p,n)$

## Atoms in the codimension poset

### Proposition

The codimension atoms in $G(r,p,n) \subset G(r,1,n)$ are

- the reflections
- *p*-connected diagonal elements (except those having non-1 eigenvalues $\zeta^c$, $\zeta^{-c}$)

Every diagonal matrix in $G(12,3,n)$ is a product of "3-connected" diagonal matrices, e.g.:

$$\mathrm{diag}(\zeta^4,\zeta^1,\zeta^1,\zeta^2,\zeta^7) = \mathrm{diag}(\zeta^4,\zeta^1,\zeta^1,1,1)\cdot\mathrm{diag}(1,1,1,\zeta^2,\zeta^7)$$

Left-hand matrix:

- element of $G(12,3,5)$ because $4+1+1+2+7$ is divisible by 3
- not 3-connected because proper subsums $4+1+1$ and $2+7$ are also divisible by 3

# Infinite family $G(r, p, n)$

## Generating cohomology from atoms in the codimension poset

Let $v_1, \ldots, v_n$ be basis for $V$ and $v_1^*, \ldots, v_n^*$ dual basis of $V^*$.

### Corollary

The cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G(r, p, n))$ is generated by $\mathrm{HH}^\bullet(S(V))$ and the following *volume forms tagged by group elements*:

- $(v_i^* - \lambda v_j^*) \otimes \mathbf{s}$, where $s$ reflects about hyperplane $v_i^* - \lambda v_j^* = 0$
- $(\bigwedge_{v_i \in (V^g)^\perp} v_i^*) \otimes \mathbf{g}$, where $g$ is $p$-connected

Example conversion of a codimension atom into a cohomology generator:

$$g = \mathrm{diag}(1, 1, 1, \zeta^2, \zeta^7) \qquad \rightsquigarrow \qquad (v_4^* \wedge v_5^*) \otimes \mathbf{g}$$

# Infinite family $G(r, p, n)$

length$\neq$codim unless $G = G(r, 1, n)$ or $G$ is Coxeter

**Corollary**

Let $G$ be a group in the family $G(r, p, n)$. The reflections are the only codimension atoms in $G$ iff $G$ is a Coxeter group or $p = 1$.

$G(r, p, n)$ with $1 < p < r$ and $n \geq 2$

$$\text{diag}(\zeta^1, \zeta^{p-1}, 1, \ldots, 1) \text{ is a codimension atom}$$

$G(r, r, n)$ with $r \geq 3$ and $n \geq 3$

$$\text{diag}(\zeta^1, \zeta^1, \zeta^{-2}, 1, \ldots, 1) \text{ is a codimension atom}$$

# Exceptional reflection groups $G_4 - G_{22}$

length≠codim in the rank two exceptional reflection groups

**Proposition**

Let $G$ be a rank two exceptional complex reflection group.

- The reflection length and codimension functions do not coincide.
- The codimension atoms are the reflections together with all elements $g$ such that $\ell(g) > \mathrm{codim}(g)$.

$G_i$ with $i \neq 8, 12$: compare orders of reflections with orders of central elements and find central $z$ in $G_i$ with $\ell(z) > \mathrm{codim}(z)$

$G_8$ and $G_{12}$: find explicit elements $g$ such that $\mathrm{codim}(gs) \geq \mathrm{codim}(g)$ for every reflection $s$ in $G$

# Exceptional reflection groups $G_{23} - G_{37}$

## Computing reflection length, atoms, and poset relations

Use CHEVIE package in GAP

Use class algebra constants (which can be computed from the irreducible characters of $G$) to avoid multiplying individual group elements

For example,

$$A \leq_{\perp} C \quad \Leftrightarrow \quad \sum_{\substack{B \\ \text{codim}(A)+\text{codim}(B)=\text{codim}(C)}} \text{ClassAlgConst}(A, B, C) \neq 0$$

# Exceptional reflection groups $G_{23} - G_{37}$

## length$\neq$codim in the non-Coxeter exceptional reflection groups

| group | # conj classes | # length$\neq$codim | # nonref atoms | dim $V$ | max ref length |
|---|---|---|---|---|---|
| 23 | 10 | 0 | 0 | 3 | 3 |
| 24 | 12 | 2 | 2 | 3 | 4 |
| 25 | 24 | 3 | 1 | 3 | 4 |
| 26 | 48 | 9 | 5 | 3 | 4 |
| 27 | 34 | 12 | 12 | 3 | 5 |
| 28 | 25 | 0 | 0 | 4 | 4 |
| 29 | 37 | 10 | 4 | 4 | 6 |
| 30 | 34 | 0 | 0 | 4 | 4 |
| 31 | 59 | 27 | 5 | 4 | 6 |
| 32 | 102 | 27 | 6 | 4 | 6 |
| 33 | 40 | 12 | 6 | 5 | 7 |
| 34 | 169 | 78 | 14 | 6 | 10 |
| 35 | 25 | 0 | 0 | 6 | 6 |
| 36 | 60 | 0 | 0 | 7 | 7 |
| 37 | 112 | 0 | 0 | 8 | 8 |

# When do reflection length and codimension coincide?

**Theorem**

Let $G$ be an irreducible complex reflection group. The reflection length and codimension functions coincide if and only if $G$ is a Coxeter group or $G = G(r, 1, n)$.

Thank you!