

Exponential Functions: Population Growth, Radioactive Decay, and More

In these examples we will use exponential and logistic functions to investigate population growth, radioactive decay, and temperature of heated objects.

Exponential Growth Models

- continuously compounded interest: $A = Pe^{rt}$
- population growth: $N(t) = N_0e^{rt}$

t = time

r = relative growth rate (a positive number)

N_0 = initial population

$N(t)$ = population after a time t has passed

Example 1. The population of a certain species of fish has a relative growth rate of 1.2% per year. It is estimated that the population in the year 2000 was 12 million.

(a) Find a function $N(t)$ that models the fish population t years since 2000.

(b) How many years will it take for the fish population to reach 17 million?

Relative Growth Rate

If you know the population at two points in time (say population N_1 at time t_1 and population N_2 at time t_2), then you can compute the relative growth rate:

$$r = \frac{\Delta \ln(N)}{\Delta t} = \frac{\ln(N_2) - \ln(N_1)}{t_2 - t_1}$$

Example 2. The population of California was 29.76 million in 1990 and 33.87 million in the year 2000. Assume the population grows exponentially.

- (a) Find the relative growth rate of the population.
- (b) Find a function that models the population of California t years since 1990.
- (c) Use the model to estimate the population of California in 2010. (Compare to the 2010 census population of 37.25 million.)

Example 3. The count in a culture of bacteria was 600 after 2 hours and 20,000 after 6 hours.

(a) Find the relative growth rate of the bacteria.

(b) Find the initial bacteria count.

(c) Find a function that models the number of bacteria $N(t)$ after t hours.

Here's a table comparing linear functions with exponential functions. The equations are different, but in both cases, you need two pieces of information to write down the equation: some kind of rate (slope or relative growth rate) and the y -intercept.

Lines	Exponential curves
equation $y = mx + b$	equation $y = y_0 e^{rx}$
slope $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$	relative growth rate $r = \frac{\Delta \ln(y)}{\Delta x} = \frac{\ln(y_2) - \ln(y_1)}{x_2 - x_1}$
y -intercept $y(0) = m \cdot 0 + b = b$	y -intercept $y(0) = y_0 \cdot e^0 = y_0 \cdot 1 = y_0$

Exponential Decay Models

- radioactive decay: $m(t) = m_0 e^{rt}$

t = time

r = decay rate (a negative number)

m_0 = initial amount of substance

$m(t)$ = amount of substance at time t

- the half-life is how long it take for an initial amount to decay to half of the initial amount (e.g. a half-life of 28 years would mean that if you started with 100 mg, then 28 years later, you would have 50 mg)
- if you know the mass at two different times (say mass m_1 at time t_1 and mass m_2 at time t_2), then you can compute the decay rate:

$$r = \frac{\Delta \ln(m)}{\Delta t} = \frac{\ln(m_2) - \ln(m_1)}{t_2 - t_1}$$

Example 4. The half-life of iodine-135 is 8 days. How long will it take for 100-mg sample to decay to a mass of 70 mg?

Example 5. If 250 mg of a radioactive element decays to 200 mg in 48 hours, find the half-life of the element.

Newton's Law of Cooling

- temperature of a heated object decreases exponentially over time; temperature of object approaches temperature of surroundings
- Newton's Law of Cooling: $T(t) = T_s + (T_0 - T_s)e^{-kt}$

t = time

k = positive constant depending on type of object

T_0 = initial temperature of object

T_s = temperature of surroundings

$T(t)$ = temperature of object at time t

Example 6. A thermometer reading 72°F is placed in a refrigerator where the temperature is a constant 38°F . After 2 minutes, the thermometer reads 60°F . What does the thermometer read after 7 minutes?

Logistic Growth Models

- population growth is generally limited by living space and food supply; logistic functions can provide a more realistic model of population growth
- logistic growth model: $P(t) = \frac{c}{1 + ae^{-bt}}$

t = time

$P(t)$ = population after time t has passed

c = carrying capacity (a positive number)

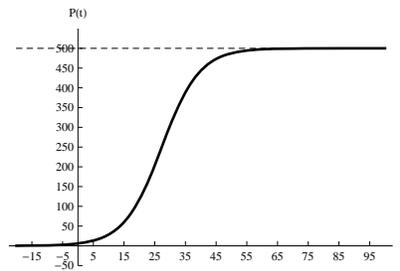
b = growth rate (a positive number)

a = shifts graph (why???)

Example 7. Suppose that six American bald eagles are captured and transported to a controlled environment where the species can regenerate its population. The population is expected to grow according to the model

$$P(t) = \frac{500}{1 + 83.33e^{-0.162t}},$$

where t is measured in years.



(a) What is the carrying capacity? What is the growth rate?

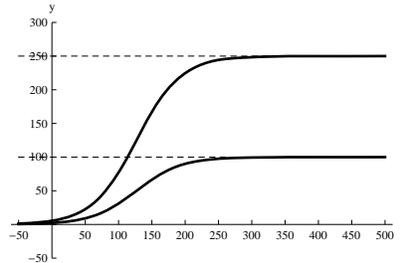
(b) When will the population be 300 eagles?

Logistic Growth Function: $P(t) = \frac{c}{1 + ae^{-bt}}$

Question: What happens when you change a , b , and c ?

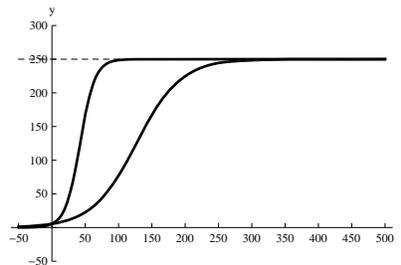
Carrying Capacity

Compare $P(t) = \frac{250}{1 + 45e^{-.03t}}$ with $N(t) = \frac{100}{1 + 45e^{-.03t}}$:



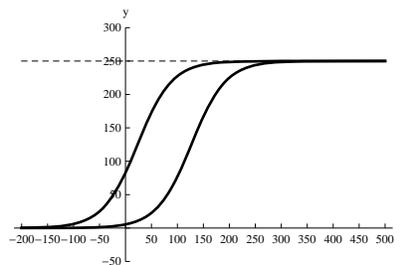
Growth Rate

Compare $P(t) = \frac{250}{1 + 45e^{-.03t}}$ with $N(t) = \frac{250}{1 + 45e^{-.09t}}$:



Shift

Compare $P(t) = \frac{250}{1 + 45e^{-.03t}}$ with $N(t) = \frac{250}{1 + 2e^{-.03t}}$:



(Note: $P(0) = \frac{250}{1 + 45} \approx 5.43$ and $N(0) = \frac{250}{1 + 2} \approx 83.33$)

Logarithmic Scales

- in chemistry, acidity of a solution is measured by hydrogen ion concentration $[H^+]$, measured in moles per liter (M)
- hydrogen ion concentrations are usually tiny numbers, expressed using negative exponents
- pH scale avoids tiny numbers and negative exponents
- logarithmic form: $pH = -\log[H^+]$
- exponential form: $[H^+] = 10^{-pH}$
- classification of solutions:
 - acidic, $pH < 7$
 - neutral, $pH = 7$
 - basic, $pH > 7$

Example 8. A sample of seawater has a hydrogen ion concentration of $[H^+] = 5.0 \times 10^{-9}M$. Calculate the pH of the sample.