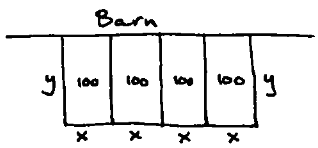
$\boxed{\text{MATH 1710.007, SPRING 2012} \\ \text{EXAM 3 REVIEW – SELECTED SOLUTIONS}}$

#1

$\boxed{\text{10b, p. 213}}$

GOAL: minimize fencing

CONSTRAINT

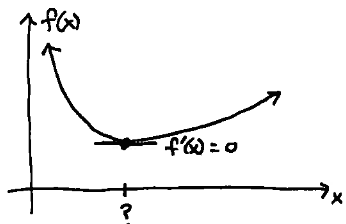


$\boxed{xy = 100}$ → $y = \frac{100}{x}$

OBJECTIVE FUNCTION (fencing)

$$f = 4x + 5y$$

$$f(x) = 4x + 5\left(\frac{100}{x}\right) = 4x + \frac{500}{x}$$



OPTIMIZATION STEP

Find $f'(x)$ and solve $f'(x) = 0$.

$$f(x) = 4x + 500x^{-1}$$

$$f'(x) = 4 - 500x^{-2} = 4 - \frac{500}{x^2}$$

$$f'(x) = 0 \longrightarrow \quad 4 = \frac{500}{x^2}$$

(cross-multiply)

$$4x^2 = 500$$

$$x^2 = 125$$

$$x = \sqrt{125} \quad \text{or} \quad \cancel{x = -\sqrt{125}}$$

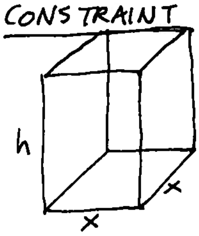
$$= 5\sqrt{5}$$

Each pen should have dimensions
$x = 5\sqrt{5}$, $y = \frac{100}{x} = \frac{100}{5\sqrt{5}} = \frac{20}{\sqrt{5}} = \frac{20\sqrt{5}}{5} = 4\sqrt{5}$
in order to minimize the amount of fencing.

$\boxed{\text{12, p. 214}}$

GOAL maximize volume

CONSTRAINT



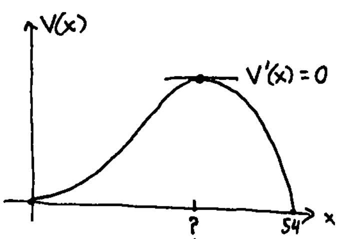
square-based box

$\boxed{2x + h = 108 \text{ in.}}$

OBJECTIVE FUNCTION (volume)

$V = x \cdot x \cdot h$ too many variables!
substitute $h = 108 - 2x$

$$V(x) = x^2(108 - 2x) = -2x^3 + 108x^2$$



OPTIMIZATION STEP

Find $V'(x)$ and solve $V'(x) = 0$.

$$V'(x) = -6x^2 + 216x$$

$$= -6x(x - 36)$$

$$V'(x) = 0 \longrightarrow \quad \cancel{x = 0} \quad \text{or} \quad x = 36$$

Use $x = 36$ and $h = 108 - 2x$ to find h:

$$h = 108 - 2(36) = 36 \text{ in.}$$

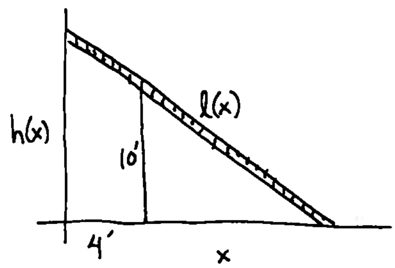
To maximize the volume, the box should be a cube with sides $x = h = 36$ in. The volume will be
$V(36) = 46\,656 \text{ in}^3$.

$\boxed{15,\ p.\ 214}$ same setup as walking/rowing problem from class

$\boxed{16,\ p.\ 214}$

GOAL minimize length of ladder

CONSTRAINT



Similar triangles!

$$\frac{10}{x} = \frac{h(x)}{4+x}$$

OBJECTIVE FUNCTION (length of ladder)

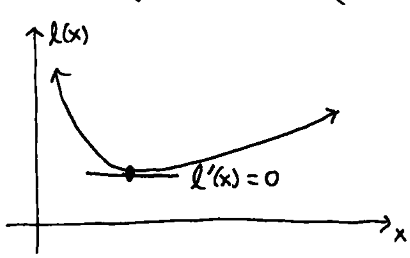
$$\ell^2 = (4+x)^2 + h^2$$

too many variables! substitute

$$h = \frac{10(4+x)}{x}$$

$$\ell(x)^2 = (4+x)^2 + \left(\frac{10(4+x)}{x}\right)^2$$

$$\ell(x) = \sqrt{(4+x)^2 + \left(\frac{10(4+x)}{x}\right)^2}$$



OPTIMIZATION STEP

Find $\ell'(x)$ and solve $\ell'(x) = 0$.

TRICK Avoid square root by doing implicit differentiation.

$$\ell(x)^2 = (4+x)^2 + h(x)^2$$

$$2\ell(x)\ell'(x) = 2(4+x) + 2h(x)h'(x)$$

$$\ell(x)\ell'(x) = (4+x) + h(x)h'(x)$$

$$\ell'(x) = \frac{(4+x) + h(x)h'(x)}{\ell(x)}$$

$$\ell'(x) = 0 \longrightarrow (4+x) + h(x)h'(x) = 0$$

Find $h'(x)$:

$$h(x) = 10 \cdot \frac{4+x}{x}$$

$$h'(x) = 10 \cdot \frac{1 \cdot x - 1(4+x)}{x^2} = \frac{-40}{x^2}$$

$$(4+x) + h(x)h'(x) = 0$$

$$\rightarrow (4+x) + \frac{10(4+x)}{x} \cdot \frac{-40}{x^2} = 0$$

$$\rightarrow (4+x) + (4+x)\left(-\frac{400}{x^3}\right) = 0$$

$$\rightarrow (4+x)\left(1 - \frac{400}{x^3}\right) = 0$$

$$\cancel{x = -4} \quad \text{or} \quad 1 = \frac{400}{x^3}$$

$$x^3 = 400$$

$$x = \sqrt[3]{400}$$

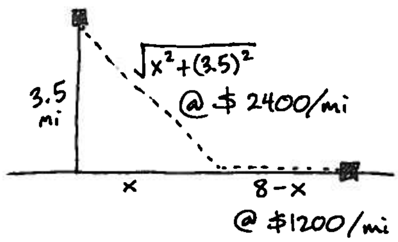
$\boxed{\text{The length of the shortest ladder is } \ell(\sqrt[3]{400}) \approx 19.2 \text{ ft.}}$

$\boxed{37,\ p.\ 216}$ Similar to can problem from class

$\boxed{31,\ p.\ 215}$

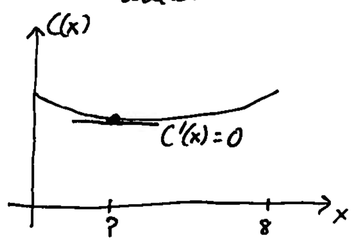
GOAL minimize cost

CONSTRAINT



OBJECTIVE FUNCTION (cost)

$$C(x) = \underbrace{2400}_{\text{cost per mi under water}}\underbrace{\sqrt{x^2+12.25}}_{\text{\# miles underwater}} + \underbrace{1200}_{\text{cost per mi underground}}\underbrace{(8-x)}_{\text{\# miles underground}}$$



OPTIMIZATION STEP

Find $C'(x)$ and solve $C'(x) = 0$.

$$C'(x) = 2400 \cdot \frac{1}{2\sqrt{x^2+12.25}} \cdot 2x - 1200$$

$$= \frac{2400x}{\sqrt{x^2+12.25}} - 1200$$

$$C'(x) = 0 \rightarrow \frac{2400x}{\sqrt{x^2+12.25}} = 1200$$

$$2400x = 1200\sqrt{x^2+12.25}$$

$$2x = \sqrt{x^2+12.25}$$

$$4x^2 = x^2 + 12.25$$

$$3x^2 = 12.25$$

$$x^2 = \frac{12.25}{3}$$

$$x = \sqrt{\frac{12.25}{3}} \quad \text{or} \quad \cancel{-\sqrt{\frac{12.25}{3}}}$$

$$\approx 2.02 \text{ mi}$$

$$8 - x \approx 5.98 \text{ mi}$$

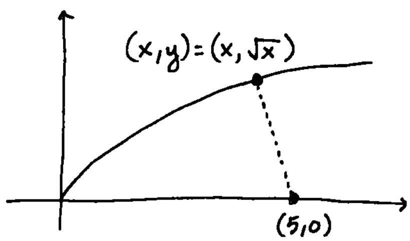
The cable should meet the shore about 5.98 mi from the power station. The min cost is $C\left(\sqrt{\frac{12.25}{3}}\right) \approx \$16874.61$

$\boxed{37,\ p.\ 216}$ similar to can problem from class

\# 2

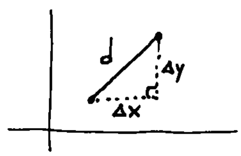
GOAL minimize distance

CONSTRAINT



OBJECTIVE FUNCTION (distance)

Recall distance formula (Pythagorean Theorem):

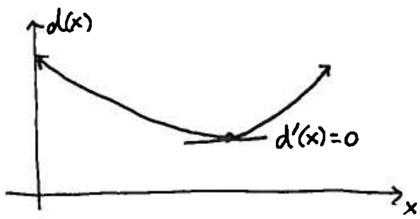


$$d^2 = \Delta x^2 + \Delta y^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

points: $(5,0)$ and $(x, \sqrt{x})$

$$d(x)^2 = (x-5)^2 + (\sqrt{x} - 0)^2$$
$$= (x-5)^2 + x$$

$$d(x) = \sqrt{(x-5)^2 + x}$$



OPTIMIZATION STEP

Find $d'(x)$ and solve $d'(x) = 0$.

TRICK Avoid $\sqrt{\ }$ by using implicit differentiation.

$$d(x)^2 = (x-5)^2 + x$$

$$2\,d(x)\,d'(x) = 2(x-5) + 1$$

$$d'(x) = \frac{2x-9}{2d(x)}$$

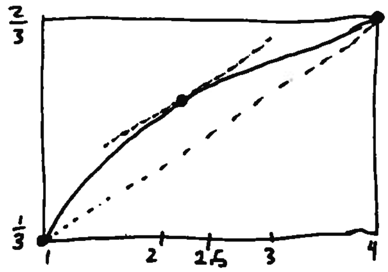
to solve fraction = 0, just solve numerator = 0

$$d'(x) = 0 \rightarrow 2x - 9 = 0$$
$$x = \frac{9}{2}$$

$$y = \sqrt{x} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

The point on the graph of $y = \sqrt{x}$ that is closest to $(5,0)$ is $\left(\frac{9}{2}, \frac{3\sqrt{2}}{2}\right)$.

#3 $f(x) = \dfrac{x}{x+2}$ on $[1,4]$



$$v_{av} = \frac{f(4)-f(1)}{4-1} = \frac{\frac{2}{3}-\frac{1}{3}}{3} = \frac{\frac{1}{3}}{3} = \frac{1}{9}$$

$$v_{inst} = f'(x) = \frac{(x+2)-x}{(x+2)^2} = \frac{2}{(x+2)^2}$$

$$v_{inst} = v_{av} \rightarrow \frac{2}{(x+2)^2} = \frac{1}{9}$$

$$(x+2)^2 = 18$$

$$x+2 = \pm\sqrt{18}$$

$$x = -2 \pm 3\sqrt{2} \qquad x = -2+3\sqrt{2} \approx 2.24$$

#4 100 meters in 9.58 seconds

$$v_{av} = \frac{100\text{ m}}{9.58\text{ s}} = \frac{\frac{1}{10}\text{ km}}{\frac{9.58}{3600}\text{ hr}} = \frac{1}{10}\cdot\frac{3600}{9.58}\text{ km/hr} \approx 37.6\text{ km/hr}$$

By the Mean Value Theorem, $v_{inst} = v_{av}$ at some point during the race, so since $v_{av} \approx 37.6$ km/hr, Bolt's speed must have exceeded 37 km/hr.

#5

- $$\lim_{\theta\to\frac{\pi}{4}} \frac{\tan\theta - \cot\theta}{\theta - \frac{\pi}{4}} \overset{\text{l'Hôpital}}{=} \lim_{\theta\to\frac{\pi}{4}} \frac{\sec^2\theta - (-\csc^2\theta)}{1}$$

  "$\frac{0}{0}$" ($\tan\frac{\pi}{4} = 1$)

  $$\overset{\text{convert to sines and cosines}}{=} \lim_{\theta\to\frac{\pi}{4}} \left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right)$$

  $$= 2 + 2 = 4$$

  $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

  $\cos^2\frac{\pi}{4} = \frac{2}{4} = \frac{1}{2}$

  $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

  $\sin^2\frac{\pi}{4} = \frac{2}{4} = \frac{1}{2}$

- $$\lim_{x\to 2} \frac{x^3 - 3x^2 + 4}{x^4 - 4x^3 + 7x^2 - 12x + 12} \overset{\text{l'Hôpital}}{=} \lim_{x\to 2} \frac{3x^2 - 6x}{4x^3 - 12x^2 + 14x - 12}$$

  "$\frac{0}{0}$"

  Trick to evaluate bottom poly $g(x) = x^4 - 4x^3 + 7x^2 - 12x + 12$

  $$\begin{array}{r|rrrrr} 2 & 1 & -4 & 7 & -12 & 12 \\ & & 2 & -4 & 6 & -12 \\ \hline & 1 & -2 & 3 & -6 & \boxed{0} \end{array}$$

  that's $g(2)$

  still "$\frac{0}{0}$"

  $$\overset{\text{l'Hôpital}}{=} \lim_{x\to 2} \frac{6x-6}{12x^2 - 24x + 14}$$

  $$= \frac{6}{14} = \frac{3}{7}$$

  $$\begin{array}{r|rrr} 2 & 4 & -12 & 14 \\ & & 8 & -8 \\ \hline & 4 & -4 & 6 \end{array}$$

  $$\begin{array}{r|rrr} 2 & 12 & -24 & 1 \\ & & 24 & \\ \hline & 12 & 0 & \end{array}$$

#5 continued

- $$\lim_{x\to 0} \frac{2\sin x - \sin(2x)}{\sin x - x\cos x} \overset{\text{l'Hop.}}{=} \lim_{x\to 0} \frac{2\cos x - 2\cos(2x)}{\cos x - (\cos x - x\sin x)} = \lim_{x\to 0} \frac{2\cos x - 2\cos(2x)}{x\sin x}$$

$$\overset{\text{l'Hop}}{=} \lim_{x\to 0} \frac{-2\sin x + 4\sin(2x)}{\sin x + x\cos x} \overset{\text{l'Hop}}{=} \lim_{x\to 0} \frac{-2\cos x + 8\cos(2x)}{\cos x + \cos x - x\sin x} = \frac{-2+8}{1+1-0} = -3$$

#6 $v(t) = \sin t + 3\cos t$, $s(0) = 4$

Find $s(t)$ with $s'(t) = v(t)$ and $s(0) = 4$.

$$s(t) = -\cos t + 3\sin t + C$$

$$s(0) = 4 \longrightarrow -\cos(0) + 3\sin(0) + C = 4$$

$$-1 + C = 4$$

$$C = 5$$

$$\boxed{s(t) = -\cos t + 3\sin t + 5}$$

#7 (solutions to even problems)

$\boxed{18, p.247}$

$$\int (3u^{-2} - 4u^2 + 1)\, du = 3\int u^{-2}\,du - 4\int u^2\,du + \int du$$

$$= 3\frac{u^{-1}}{-1} - 4\frac{u^3}{3} + u + C$$

$$= -\frac{3}{u} - \frac{4}{3}u^3 + u + C$$

$\boxed{20, p.247}$

$$\int \left(\frac{5}{t^2} + 4t^2\right) dt$$

$$= 5\int t^{-2}dt + 4\int t^2\,dt$$

$$= 5\frac{t^{-1}}{-1} + 4\frac{t^3}{3} + C$$

$$= -\frac{5}{t} + \frac{4}{3}t^3 + C$$

$\boxed{22, p.247}$

$$\int 5m(12m^3 - 10m)\, dm$$

$$= \int (60m^4 - 50m^2)\, dm$$

$$= 60\int m^4\,dm - 50\int m^2\,dm$$

$$= 60\frac{m^5}{5} - 50\frac{m^3}{3} + C$$

$$= 12m^5 - \frac{50}{3}m^3 + C$$

$\boxed{24, p.247}$

$$\int 6\sqrt[3]{x}\, dx$$

$$= 6\int x^{1/3}\, dx$$

$$= 6\,\frac{3}{4}\,x^{4/3} + C$$

$$= \frac{9}{2}x^{4/3} + C$$

$\boxed{26, p.247}$

$$\int \left(\sin 4t - \sin\left(\frac{t}{4}\right)\right) dt$$

$$= \int \sin 4t\, dt - \int \sin\left(\frac{t}{4}\right) dt$$

$$= \left(-\frac{\cos 4t}{4}\right) - \left(-4\cos\left(\frac{t}{4}\right)\right) + C$$

$$= -\frac{\cos 4t}{4} + 4\cos\left(\frac{t}{4}\right) + C$$

#7 continued

$\boxed{28, p.247}$ $\int 2\sec^2 2v\,dv = \tan 2v + C$

$\boxed{30, p.247}$

$$\int \frac{\sin\theta - 1}{\cos^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos^2\theta}\,d\theta - \int \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}\,d\theta - \int \sec^2\theta\,d\theta$$

$$= \int \sec\theta\tan\theta\,d\theta - \int \sec^2\theta\,d\theta$$

$$= \sec\theta - \tan\theta + C$$

#9

Lowering Powers: $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

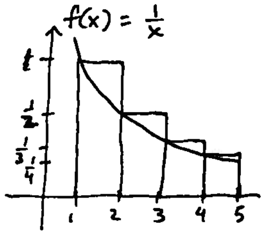
- $$\int \sin^2 t\,dt = \int \frac{1-\cos 2t}{2}\,dt = \int \frac{1}{2}\,dt - \frac{1}{2}\int \cos 2t\,dt$$
  $$= \frac{1}{2}t - \frac{1}{2}\left(\frac{1}{2}\sin 2t\right) + C$$
  $$= \frac{1}{2}t - \frac{1}{4}\sin 2t + C$$
- $$\int \cos^2 t\,dt = \frac{1}{2}t + \frac{1}{4}\sin 2t + C$$
- $$\int \sin t\cos t\,dt = \frac{1}{2}\int \sin 2t\,dt = \frac{1}{2}\left(-\frac{\cos 2t}{2}\right) + C$$
  $$= -\frac{\cos 2t}{4} + C$$

  DOUBLE ANGLE
  $\sin 2t = 2\sin t\cos t$
- $$\int \tan^2 t\,dt = \int (\sec^2 t - 1)\,dt = \tan t - t + C$$

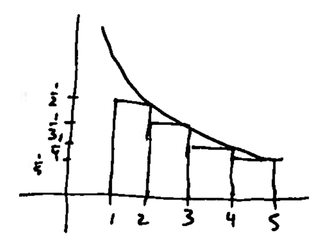
  PYTHAGOREAN
  $$\frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$
  $$\tan^2 t + 1 = \sec^2 t$$

#11 $\boxed{16, p.261}$

Left Riemann sum: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.08$



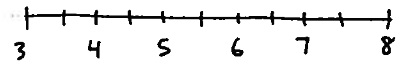
Right Riemann sum: $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} \approx 1.28$



#12 $v(t) = \sqrt{7t}$

(a)

time interval	sample time	sample velocity	change in time	approx. disp.
[3, 3.5]	3	$\sqrt{21}$	.5	2.291
[3.5, 4]	3.5	$\sqrt{24.5}$	.5	2.475
[4, 4.5]	4	$\sqrt{28}$	.5	2.646
[4.5, 5]	4.5	$\sqrt{31.5}$	.5	2.806
[5, 5.5]	5	$\sqrt{35}$	.5	2.958
[5.5, 6]	5.5	$\sqrt{38.5}$	.5	3.102
[6, 6.5]	6	$\sqrt{42}$	.5	3.240
[6.5, 7]	6.5	$\sqrt{45.5}$	.5	3.373
[7, 7.5]	7	7	.5	3.5
[7.5, 8]	7.5	$\sqrt{52.5}$	.5	3.623
				$\boxed{30.014 \text{ m}}$



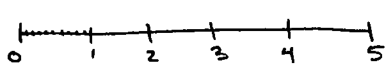
$$= \sum_{K=0}^{9} v(3 + K(.5)) \cdot .5 = \sum_{K=0}^{9} \sqrt{7(3+.5K)}\,(.5)$$

(b)

time interval	sample time	sample velocity	change in time	approx. disp.
[3, 3.5]	3.5	$\sqrt{24.5}$	.5	2.475
[3.5, 4]	4	$\sqrt{28}$	.5	2.646
[4, 4.5]	4.5	$\sqrt{31.5}$	.5	2.806
[4.5, 5]	5	$\sqrt{35}$	.5	2.958
[5, 5.5]	5.5	$\sqrt{38.5}$	.5	3.102
[5.5, 6]	6	$\sqrt{42}$	.5	3.240
[6, 6.5]	6.5	$\sqrt{45.5}$	.5	3.373
[6.5, 7]	7	7	.5	3.5
[7, 7.5]	7.5	$\sqrt{52.5}$	.5	3.623
[7.5, 8]	8	$\sqrt{56}$	.5	3.742
				$\boxed{31.465}$

$$= \sum_{K=1}^{10} v(3 + .5K) .5 = \sum_{K=1}^{10} \sqrt{7(3+.5K)}\,(.5)$$

#13



$$\sum_{K=0}^{49} v\left(\frac{K}{10}\right) \cdot \frac{1}{10} = \sum_{K=0}^{49} \sqrt{\frac{7K}{10}} \cdot \frac{1}{10}$$

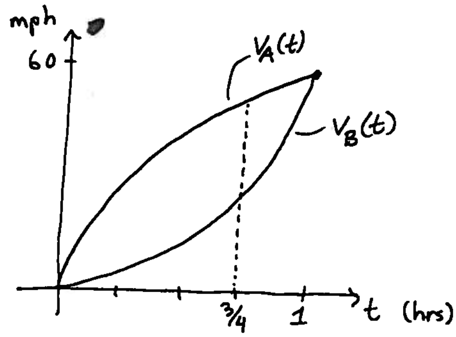
$\boxed{31-34, \ p. 277}$

#19 $\int_0^a f(x)\,dx = 16, \quad \int_0^b f(x)\,dx = 16-5 = 11, \quad \int_a^c f(x)\,dx = -5+11 = 6,$

$\int_0^0 f(x)\,dx = 16-5+11 = 22$

#15 $V_A(t) = 60\sqrt{t}, \quad V_B(t) = 60t^2, \quad 0 \le t \le 1$

(a)



(b) car A goes faster

(c) How far has car A gone?

$$\int_0^1 60\sqrt{t}\,dt = 60\int_0^1 t^{1/2}\,dt$$

$$= 60 \cdot \frac{2}{3} t^{3/2} \Big|_0^1 = 40 \text{ mi}$$

How far has car B gone?

$$\int_0^1 60t^2\,dt = 60\int_0^1 t^2\,dt$$

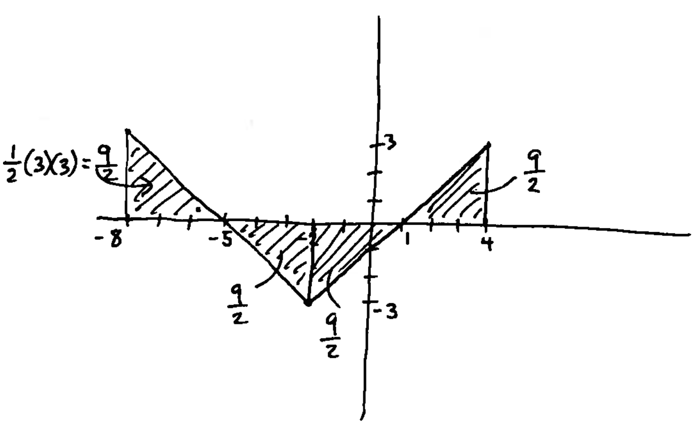
$$= 60\,\frac{t^3}{3}\Big|_0^1 = 20t^3\Big|_0^1 = 20 \text{ mi}$$

Car A is 20 miles ahead of car B.

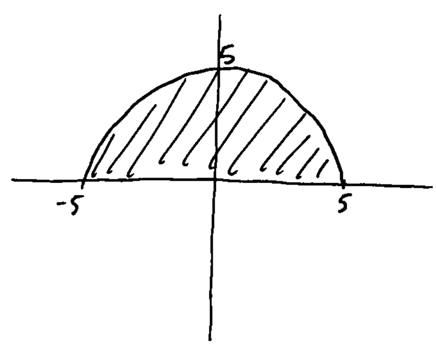
#17

- $\int_{-8}^{4} (|x+2| - 3)\,dx = \frac{9}{2} - \frac{9}{2} - \frac{9}{2} + \frac{9}{2} = 0$

  $|x|$ shifted left 2, down 3



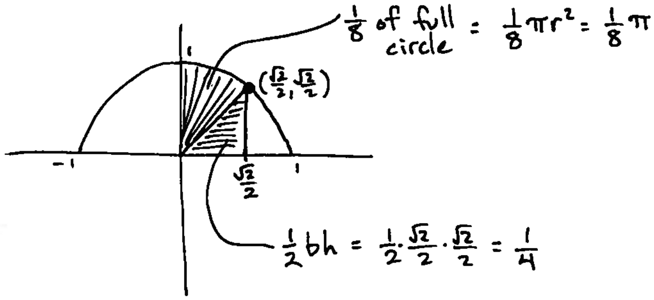
- $\int_{-5}^{5} \sqrt{25-x^2}\,dx = \frac{25\pi}{2}$



area of semicircle $\frac{1}{2}\pi r^2$

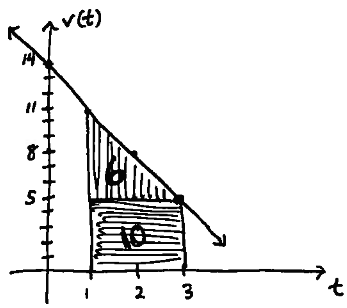
#17 continued

- $$\int_0^{\frac{\sqrt{2}}{2}} \sqrt{1-x^2}\,dx = \frac{\pi}{8} + \frac{1}{4} = \frac{\pi+2}{8}$$



#20

$v(t) = 14 - 3t$



$$A_1(x) = \int_1^x v(t)\,dt$$

$A_1(1) = 0$

$A_1(3) = 10 + 6 = 16$

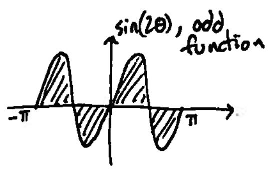
#21

$$\frac{d}{dx}\int_0^x \frac{3t^7 - \sin t}{\sqrt{t^2+16}}\,dt = \frac{3x^7 - \sin x}{\sqrt{x^2+16}}$$

FUNDAMENTAL THEOREM OF CALCULUS

#22

- $$\int_{-\pi}^{\pi} \sin(2\theta)\,d\theta = 0$$



- $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta$$

$$= 2\sin\theta\Big|_0^{\frac{\pi}{2}} = 2\left(\sin\frac{\pi}{2} - \sin 0\right) = 2$$

- $$\int_{-1}^{1}(4x^5 - 5x^4 + 3x - 15)\,dx$$

$$= \underbrace{\int_{-1}^{1}(4x^5+3x)\,dx}_{\text{odd function}} + \underbrace{\int_{-1}^{1}(-5x^4-15)\,dx}_{\text{even function}}$$

$$= 0 + 2\int_0^1(-5x^4 - 15)\,dx$$

$$= -10\int_0^1(x^4+3)\,dx$$

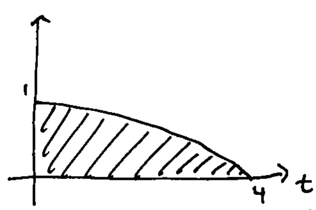
$$= -10\left(\frac{x^5}{5} + 3x\right)\Big|_0^1$$

$$= -10\left(\left(\frac{1}{5}+3\right) - 0\right)$$

$$= -\frac{10}{5} - 30 = -32$$

#23

$v(t) = 1 - \frac{1}{16}t^2, \quad 0 \le t \le 4$



$$\text{displacement} = \int_0^4\left(1 - \frac{1}{16}t^2\right)dt$$

$$= \int_0^4 1\,dt - \frac{1}{16}\int_0^4 t^2\,dt$$

$$= t\Big|_0^4 - \frac{1}{16}\frac{t^3}{3}\Big|_0^4$$

$$= (4-0) - \frac{1}{16}\left(\frac{4^3}{3} - 0\right)$$

$$= 4 - \frac{4}{3} = \frac{12}{3} - \frac{4}{3} = \frac{8}{3}$$

$$V_{av} = \frac{\frac{8}{3}}{4-0} = \frac{8}{12} = \frac{2}{3} \text{ units/s}$$