Math 1710.007, Spring 2012
Exam 2 Review – Selected Solutions

#2 $f(x) = x^5 - 2x^3 - 10x + \pi^5$ constant!

$f'(x) = 5x^4 - 6x^2 - 10$

$f''(x) = 20x^3 - 12x$

#3 PRODUCT RULE

• $y = (3x+1)(5x-4)$

$y' = 3(5x-4) + 5(3x+1)$

$= 15x - 12 + 15x + 5$

$= 30x - 7$

• $y = x \sin x$

$y' = 1 \cdot \sin x + \cos x \cdot x$

$= \sin x + x\cos x$

• $y = \underbrace{\sqrt{x}(\sqrt{x}+1)}_{f(x)}\underbrace{(\sqrt{x}+2)}_{g(x)}$

$f'(x) = \frac{1}{2\sqrt{x}}(\sqrt{x}+1) + \frac{1}{2\sqrt{x}} \cdot \sqrt{x}$

$= \frac{1}{2} + \frac{1}{2\sqrt{x}} + \frac{1}{2} = 1 + \frac{1}{2\sqrt{x}}$

$g'(x) = \frac{1}{2\sqrt{x}}$

$y' = f'(x)g(x) + g'(x)f(x)$

$= \left(1 + \frac{1}{2\sqrt{x}}\right)(\sqrt{x}+2) + \frac{1}{2\cancel{\sqrt{x}}}\cancel{\sqrt{x}}(\sqrt{x}+1)$

$= \sqrt{x} + 2 + \frac{1}{2\sqrt{x}} \cdot \sqrt{x} + \frac{1}{2\sqrt{x}} \cdot 2 + \frac{1}{2}\sqrt{x} + \frac{1}{2}$

$= \sqrt{x} + 2 + \frac{1}{2} + \frac{1}{\sqrt{x}} + \frac{1}{2}\sqrt{x} + \frac{1}{2}$

$= \frac{1}{\sqrt{x}} + 3 + \frac{3}{2}\sqrt{x}$

#4 QUOTIENT RULE

• $y = \frac{1}{1+x^2}$

$y' = \frac{0 \cdot (1+x^2) - 2x \cdot 1}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$

• $y = \frac{3x+2}{10x+7}$

$y' = \frac{3(10x+7) - 10(3x+2)}{(10x+7)^2}$

$= \frac{30x + 21 - 30x - 20}{(10x+7)^2}$

$= \frac{1}{(10x+7)^2}$

• $y = \frac{\sin x}{x+1}$

$y' = \frac{\cos x\,(x+1) - 1 \cdot \sin x}{(x+1)^2} = \frac{x\cos x + \cos x - \sin x}{(x+1)^2}$

#5 $f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -1x^{-2} = -\frac{1}{x^2}$

$f''(x) = 2x^{-3} = \frac{2}{x^3}$

$f'''(x) = -6x^{-4} = -\frac{6}{x^4}$

$f^{(4)}(x) = 24x^{-5} = \frac{24}{x^5}$

#6 pattern starts over every multiple of 4

$f(x) = \sin x$	$f^{(4)}(x) = \sin x$	$f^{(36)}(x) = \sin x$
$f'(x) = \cos x$	$f^{(5)}(x) = \cos x$	$f^{(37)}(x) = \cos x$
$f''(x) = -\sin x$	$\vdots$	$f^{(38)}(x) = -\sin x$
$f'''(x) = -\cos x$		$f^{(39)}(x) = -\cos x$

#7 SEE NOTES FROM CLASS

#8 • $y = \frac{\sin x}{1+\cos x}$

$y' = \frac{\cos x(1+\cos x) - (-\sin x)(\sin x)}{(1+\cos x)^2}$

$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$

PYTHAGOREAN ID $\sin^2 x + \cos^2 x = 1$

$= \frac{\cos x + 1}{(1+\cos x)^2}$

$= \frac{1}{1+\cos x}$

• $y = \sec x \tan x$

$y' = (\sec x \tan x)\tan x + \sec^2 x \cdot \sec x$

$= \sec x \tan^2 x + \sec x \sec^2 x$

$= \sec x\,(\tan^2 x + \sec^2 x)$

$= \sec x\,(1 + 2\tan^2 x)$ PYTHAGOREAN ID $1 + \tan^2 x = \sec^2 x$

- $y = \csc^2 x = (\csc x)^2$

$y' = 2\csc x(-\csc x \cot x)$

$= -2\csc^2 x \cot x$

- $y = \sec x \cot x = \frac{1}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{\sin x} = \csc x$

$y' = -\csc x \cot x$

#9
- $y = \sqrt{x^4 + x^2 + 1}$

$y' = \frac{1}{2\sqrt{x^4 + x^2 + 1}}(4x^3 + 2x)$

$= \frac{4x^3 + 2x}{2\sqrt{x^4 + x^2 + 1}} = \frac{2x^3 + x}{\sqrt{x^4 + x^2 + 1}}$

CHAIN RULE

- $y = \sec(x^3 + 5)$

$y' = \sec(x^3+5)\tan(x^3+5)(3x^2)$

$= 3x^2 \sec(x^3+5)\tan(x^3+5)$

- $y = \cos^9 x = (\cos x)^9$

$y' = 9(\cos x)^8(-\sin x)$

$= -9\cos^8 x \sin x$

- $y = (5x+1)^{100}$

$y' = 100(5x+1)^{99} \cdot 5$

$= 500(5x+1)^{99}$

#10
- $y = \cos^2(3x) = (\cos(3x))^2$

inside: $\cos(3x)$

outside: $(\quad)^2$

$y' = 2(\cos(3x)) \frac{d}{dx}[\cos(3x)]$

$= 2\cos(3x)(-3\sin(3x))$

$= -6\sin(3x)\cos(3x)$

CHAIN RULE

$$\boxed{\begin{array}{l} \frac{d}{dx}[\cos(3x)] = -\sin(3x) \cdot 3 \\ \text{inside: } 3x \\ \text{outside: } \cos(\ ) \end{array}}$$

- $y = \sin(\sqrt{x^4+1})$

inside: $\sqrt{x^4+1}$

outside: $\sin(\quad)$

$y' = \cos(\sqrt{x^4+1}) \cdot \frac{d}{dx}[\sqrt{x^4+1}]$

$= \cos(\sqrt{x^4+1}) \cdot \frac{1}{2\sqrt{x^4+1}} \cdot 4x^3$

$= \frac{2x^3\cos(\sqrt{x^4+1})}{\sqrt{x^4+1}}$

#11
- $y = \underbrace{x^3}_{f(x)}\underbrace{\cos(5x^2)}_{g(x)}$

$f'(x) = 3x^2$

$g'(x) = -\sin(5x^2)(10x)$

$= -10x\sin(5x^2)$

$y' = 3x^2\cos(5x^2) - 10x\sin(5x^2) \cdot x^3$

$= 3x^2\cos(5x^2) - 10x^4\sin(5x^2)$

- $y = \underbrace{(2x-1)^3}_{f(x)}\underbrace{(7-x^5)^4}_{g(x)}$

$f'(x) = 3(2x-1)^2 \cdot 2 = 6(2x-1)^2$

$g'(x) = 4(7-x^5)^3(-5x^4) = -20x^4(7-x^5)^3$

$y' = 6(2x-1)^2(7-x^5)^4 - 20x^4(7-x^5)^3(2x-1)^3$

$= 2(2x-1)^2(7-x^5)^3[3(7-x^5) - 10x^4(2x-1)]$

$= 2(2x-1)^2(7-x^5)^3(21 - 3x^5 - 20x^5 + 10x^4)$

$= 2(2x-1)^2(7-x^5)^3(21 + 10x^4 - 23x^5)$

#12 $f(t) = -16t^2 + 64t + 32$

(a) $v(t) = f'(t) = -32t + 64$

(b) highest point is when $v(t) = 0$

$-32t + 64 = 0$

$-t + 32 = 0$

$t = 32$ seconds

(c) stone hits ground when position $f(t) = 0$.

$-16t^2 + 64t + 32 = 0$

$t^2 - 4t - 2 = 0$

$t = \frac{4 \pm \sqrt{16+8}}{2} = \frac{4 \pm \sqrt{24}}{2}$

$= \frac{4 \pm 2\sqrt{6}}{2}$

$= 2 \pm \sqrt{6}$

#12 (c) cont.

$t = 2+\sqrt{6}$ or $\cancel{t = 2-\sqrt{6}}$

≈ 4.45 sec $\quad \cancel{\approx -.45}$

(d) Plug $t = 2+\sqrt{6} \approx 4.45$ sec into velocity function.

$$v(2+\sqrt{6}) = -32(2+\sqrt{6}) + 64$$
$$\approx -78.38 \text{ ft/sec}$$

#13 $\quad x^3 + y^3 = 2xy$

(a) $\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 2\frac{d}{dx}(xy)$

$$3x^2 + 3y^2y' = 2[1\cdot y + y'x]$$
$$3x^2 + 3y^2y' = 2y + 2xy'$$
$$3y^2y' - 2xy' = 2y - 3x^2$$
$$y'(3y^2 - 2x) = 2y - 3x^2$$
$$y' = \frac{2y - 3x^2}{3y^2 - 2x}$$

(b) $(1)^3 + (1)^3 \stackrel{?}{=} 2(1)(1)$ ✓

(c) $\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{2(1) - 3(1)^2}{3(1)^2 - 2(1)} = \frac{-1}{1} = -1$

point: $(1,1)$

slope: -1

tangent line: $y = \underset{\text{adjust slope}}{-}\underbrace{(x-1)}_{\text{shift right 1}}\underbrace{+1}_{\text{up 1}}$

#14

3.8.9

volume and radius change over time

① Eqn relating volume and radius

$$V = \frac{4}{3}\pi r^3$$

$$\boxed{V(t) = \frac{4}{3}\pi\, r(t)^3}$$ [V and r are functions of time]

② Differentiate with respect to time to get related rates eqn:

$$V'(t) = \frac{4}{3}\pi \cdot \underbrace{3r(t)^2\, r'(t)}_{\text{CHAIN RULE}}$$

$$\boxed{V'(t) = 4\pi\, r(t)^2\, r'(t)}$$

③ Plug in known values/solve for unknown

$\boxed{\begin{array}{l} V'(t) = 15 \text{ in}^3/\text{min} \\ r(t) = 10 \text{ in} \\ r'(t) = ? \end{array}}$

$$15 = 4\pi \cdot 100\, r'(t)$$
$$r'(t) = \frac{15}{400\pi} = \frac{3}{80\pi} \text{ in/min}$$
$$\boxed{r'(t) \approx .0119 \text{ in/min}}$$

3.8.22

depth and volume change over time

① Eqn relating depth and volume

$$V = \pi r^2 d$$

$$\boxed{V(t) = 4\pi\, d(t)}$$ [r = 2 in always, V and d depend on time]

② Differentiate with respect to time to get related rates eqn:

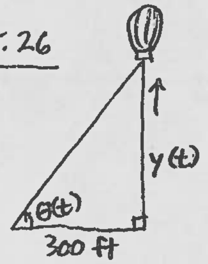
$$\boxed{V'(t) = 4\pi\, d'(t)}$$

③ Plug in known values/solve for unknown

$\boxed{\begin{array}{l} V'(t) = ? \\ d'(t) = .25 \text{ in/s} \end{array}}$

$$V'(t) = 4\pi(.25) \text{ in}^3/\text{s}$$
$$\boxed{V'(t) \approx 3.14 \text{ in}^3/\text{s}}$$

3.8.26



angle of elevation and height of balloon change over time

① Eqn relating angle of elevation and height of balloon

$$\tan[\theta(t)] = \frac{y(t)}{300}$$

$$300\tan[\theta(t)] = y(t)$$

② Differentiate with respect to time to get related rates eqn:

$$\underbrace{300\sec^2[\theta(t)]\cdot\theta'(t)}_{\text{CHAIN RULE}} = y'(t)$$

$$\boxed{\frac{300\,\theta'(t)}{\cos^2[\theta(t)]} = y'(t)}$$

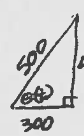
③ Plug in known values/solve for unknown

$y'(t) = 20$ ft/s

$\theta'(t) = ?$

$y(t) = 400$ ft

$\cos\theta(t) = \frac{300}{500} = \frac{3}{5}$

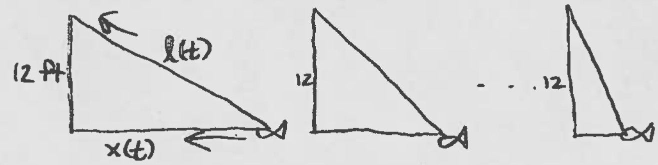


$$300\,\theta'(t) = 20\left(\frac{3}{5}\right)^2$$

$$\theta'(t) = \frac{180}{300(25)}$$

$$\boxed{\theta'(t) = .024 \text{ rad/s}}$$

3.8.29



length of fishing line and distance from fish to fisherman change over time

① Eqn relating length of fishing line and distance between fish and fisherman

$$x(t)^2 + l(t)^2 = 12^2$$

$$x(t)^2 + l(t)^2 = 144$$

② Differentiate with respect to time to get related rate eqn:

$$2\,x(t)\,x'(t) + 2\,l(t)\,l'(t) = 0$$

$$\boxed{x(t)\,x'(t) + l(t)\,l'(t) = 0}$$

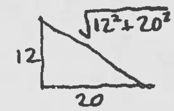
③ Plug in known values/solve for unknown

$x(t) = 20$ ft.

$l(t) = 4\sqrt{34}$

$x'(t) = ?$

$l'(t) = 4$ in/s



$$20\,x'(t) + 4\sqrt{34}\cdot 4 = 0$$

$$x'(t) = -\frac{16\sqrt{34}}{20} \text{ in/s}$$

The distance between the fish and fisherman is decreasing at a rate of about 4.66 in/s.

#17 $f(x) = (x+1)(2x^2 - 17x + 41) = 2x^3 - 17x^2 + 41x$

$\qquad + 2x^2 - 17x + 41$

$$\boxed{f(x) = 2x^3 - 15x^2 + 24x + 41}$$

(a) x-ints: solve $f(x) = 0$

$x + 1 = 0$ or $2x^2 - 17x + 41 = 0$

$x = -1$ no real sol'ns since $b^2 - 4ac$ is negative

y-int: set $x = 0$

$f(0) = 41$

$\boxed{\text{x-int: } (-1,0) \quad \text{y-int: } (0,41)}$

(b) end behavior:

leading term $2x^3$ has basic shape ,

so $\lim\limits_{x \to -\infty} f(x) = -\infty$, $\lim\limits_{x \to \infty} f(x) = \infty$

(c) above or below x-axis?

Analyze sign chart for f

$f(x) = (x+1)(2x^2 - 17x + 41)$

sign of...		-1	
$x+1$	$-$		$+$
$2x^2 - 17x + 41$	$+$		$+$
$f(x)$	$-$		$+$

below x-axis on $(-\infty, -1)$

above x-axis on $(-1, \infty)$

(d) increasing or decreasing?

Analyze sign chart for f'

$$f'(x) = 6x^2 - 30x + 24$$

$$= 6(x^2 - 5x + 4)$$

$$= 6(x-1)(x-4) \quad \leftarrow \text{zeros: } x=1,\ x=4$$

sign of...		1		4	
$6(x-1)$	$-$		$+$		$+$
$(x-4)$	$-$		$-$		$+$
$f'(x)$	$+$		$-$		$+$
		↑ switch sign		↑ switch sign	
		$f(1) = 52$		$f(4) = 25$	

increasing on $(-\infty, 1)$ and $(4, \infty)$

decreasing on $(1, 4)$

local max at $(1, 52)$

local min at $(4, 25)$

Trick for evaluating poly's: set up like you're going to do synthetic division:

4	2	-15	24	41
		8	-28	-16
	2	-7	-4	$\boxed{25}$

that's $f(4)$!

(e) concave up or down?

Analyze sign chart for f''

$$f''(x) = 12x - 30 = 6(2x-5) \leftarrow \text{zeros: } x = \tfrac{5}{2}$$

		$\frac{5}{2}$	
$f''(x)$	$-$		$+$

$\uparrow$ switch concavity

$\boxed{\text{concave up on } \left(\tfrac{5}{2}, \infty\right)}$

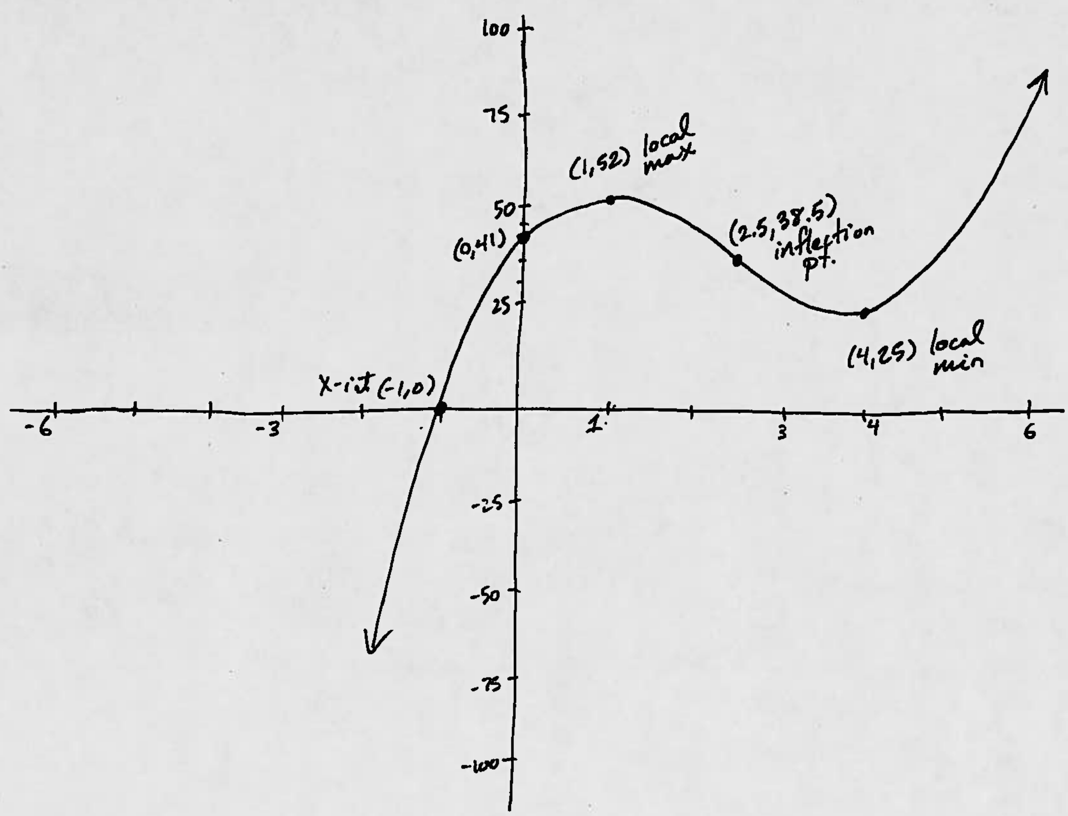
$\boxed{\text{concave down on } \left(-\infty, \tfrac{5}{2}\right)}$

$\frac{5}{2}$	2	-15	24	41
		5	-25	$-\frac{5}{2}$
	2	-10	-1	$\boxed{\frac{77}{2}}$

$$f\left(\tfrac{5}{2}\right) = \frac{77}{2} = 38.5$$

$\boxed{\text{inflection point at } (2.5, 38.5)}$

(f) graph!



#18 $f(x) = \dfrac{x}{x^2+1}$

ⓐ x-ints: solve $f(x) = 0$

(for a fraction, can just solve numerator = 0)

$x = 0$

y-int: set $x = 0$

$f(0) = 0$

$\boxed{\text{x- and y-int } (0,0)}$

vertical asymptotes: solve denominator = 0

(this is where f is undefined)

$x^2+1 = 0$

no real sol'ns, so no vert. asymptotes

ⓑ end behavior

$\dfrac{\text{small deg}}{\text{large deg}} \rightarrow \boxed{y=0 \text{ is horizontal asymptote}}$

ⓒ above or below x-axis?

Analyze sign chart for f

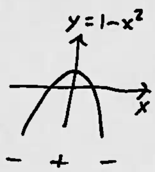
		0	
x	$-$		$+$
x^2+1	$+$		$+$
f	$-$		$+$

below x-axis on $(-\infty, 0)$
above x-axis on $(0, \infty)$

ⓓ increasing or decreasing?

Analyze sign chart for f'

$$f'(x) = \frac{x^2+1 - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}$$



$\rightarrow$

		-1		1	
$1-x^2$	$-$		$+$		$-$
$(x^2+1)^2$	$+$		$+$		$+$
$f'(x)$	$-$		$+$		$-$

increasing on $(-1, 1)$
decreasing on $(-\infty, -1)$ and $(1, \infty)$

$f(-1) = -\frac{1}{2}$

$f(1) = \frac{1}{2}$

local min at $(-1, -\frac{1}{2})$
local max at $(1, \frac{1}{2})$

(e) concave up or down?

Analyze sign chart for f''

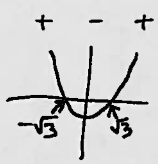
$$f''(x) = \frac{(-2x)(x^2+1)^2 - 2(x^2+1)(2x)(1-x^2)}{(x^2+1)^4}$$

$$= \frac{-2x(x^2+1) - 4x(1-x^2)}{(x^2+1)^3} = \frac{-2x^3 - 2x + 4x^3 - 4x}{(x^2+1)^3}$$

$$= \frac{2x^3 - 6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

← zeros: $x = 0, \pm\sqrt{}$

← zeros: none



→

	$-\sqrt{3}$		0		$\sqrt{3}$	
$2x$	$-$		$-$	$+$		$+$
x^2-3	$+$		$-$	$-$		$+$
$(x^2+1)^3$	$+$		$+$	$+$		$+$
$f''(x)$	$-$		$+$	$-$		$+$

↑ switch concavity ↑ ↑ switch concavity

$f(0) = 0$

$f(\sqrt{3}) = \frac{\sqrt{3}}{4}$

$f(-\sqrt{3}) = -\frac{\sqrt{3}}{4}$

concave up on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty$

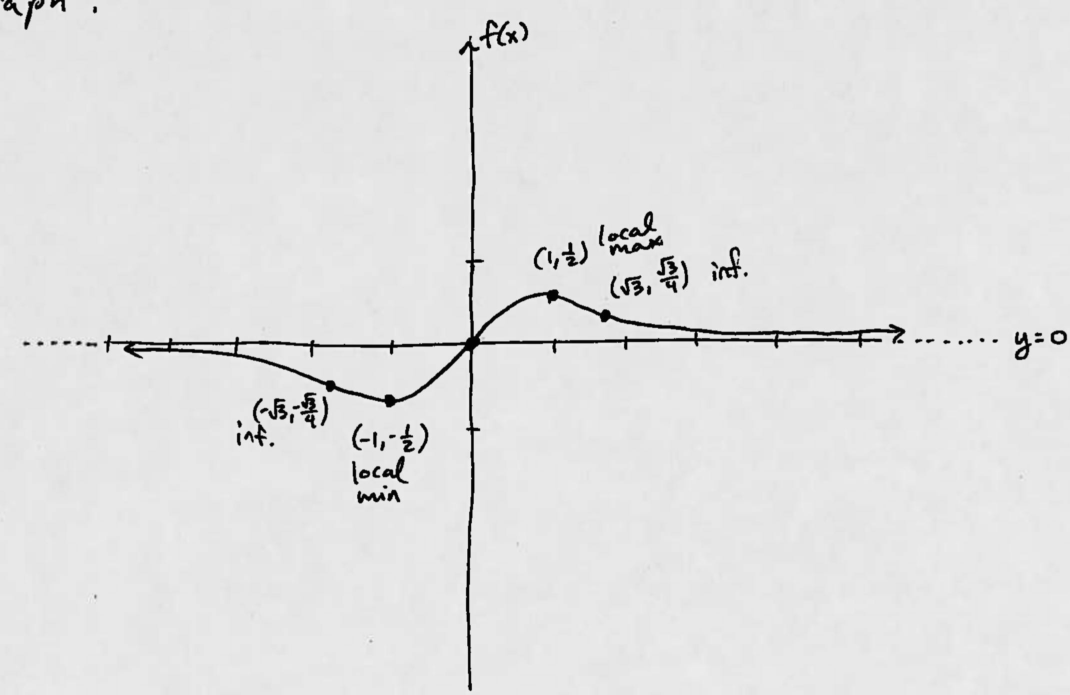
concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3}$

inflection points at $(0,0)$, $(\sqrt{3}, \frac{\sqrt{3}}{4})$, $(-\sqrt{3}, -\frac{\sqrt{3}}{4})$

$\sqrt{3} \approx 1.73$

$\frac{\sqrt{3}}{4} \approx .433$

(f) graph!



#19

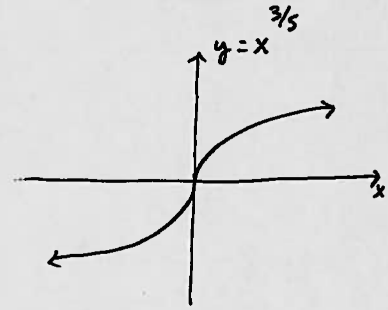
$y = x^{3/5}$

$y' = \frac{3}{5} x^{-2/5} = \frac{3}{5x^{2/5}}$

$y'' = -\frac{6}{25} x^{-7/5} = \frac{-6}{25x^{7/5}}$

0

y	$-$	$+$
y'	$+$	$+$
y''	$+$	$-$
	inc, concave up	inc, concave down



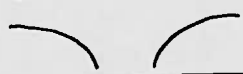
$y = x^{4/5}$

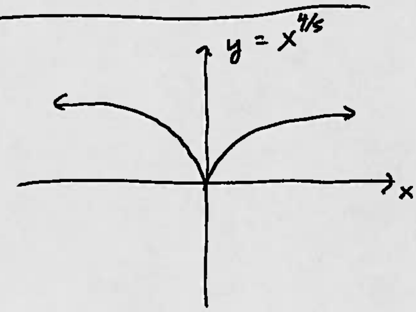
$y' = \frac{4}{5} x^{-1/5} = \frac{4}{5x^{1/5}}$

$y'' = -\frac{4}{25} x^{-6/5} = -\frac{4}{25x^{6/5}}$

0

y	$+$	$+$
y'	$-$	$+$
y''	$-$	$-$
	dec concave down	inc concave down





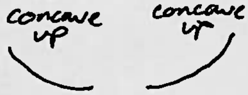
$y = x^{6/5}$

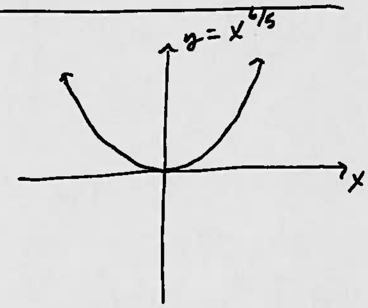
$y' = \frac{6}{5} x^{1/5}$

$y'' = \frac{6}{25} x^{-4/5} = \frac{6}{25x^{4/5}}$

0

y	$+$	$+$
y'	$-$	$+$
y''	$+$	$+$
	dec concave up	inc concave up





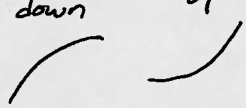
$y = x^{7/5}$

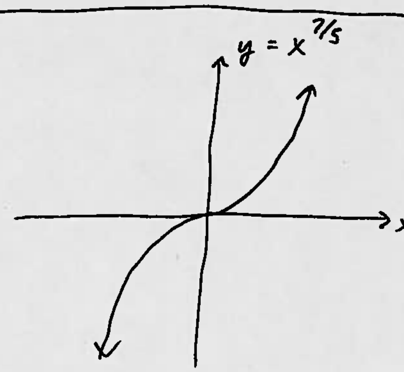
$y' = \frac{7}{5} x^{2/5}$

$y'' = \frac{14}{25} x^{-3/5} = \frac{14}{25x^{3/5}}$

0

y	$-$	$+$
y'	$+$	$+$
y''	$-$	$+$
	inc, concave down	inc concave up





#20 $f(x) = x^3 - 12x$ on $[0,4]$

$f'(x) = 3x^2 - 12 = 3(x^2-4)$
$= 3(x+2)(x-2)$

CRITICAL NUMBERS (for f')

$x = 2$, $\cancel{-2}$

ignore because -2 is not in the interval $[0,4]$

Table of values

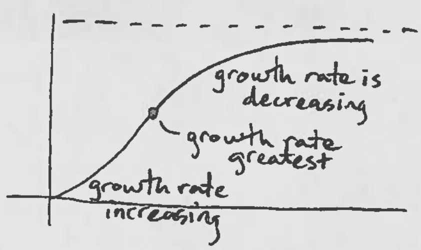
	x	$f(x)$	
crit. num. →	2	−16	← smallest y-value
endpoints →	0	0	
endpoints →	4	16	← biggest y-value

abs. max point: $(4,16)$

abs. min point: $(2,-16)$

#21 $P(t) = \dfrac{1500t^2}{2t^2+3}$

(a)



(b) $P(t) = 1500 \cdot \dfrac{t^2}{2t^2+3}$

$$P'(t) = 1500 \cdot \left[\frac{2t(2t^2+3) - 4t \cdot t^2}{(2t^2+3)^2}\right]$$

$$= 1500\left[\frac{6t}{(2t^2+3)^2}\right] = 9000 \cdot \frac{t}{(2t^2+3)^2}$$

(c) the growth rate is greatest when there is an inflection point on the graph, i.e. when $P''(t) = 0$

$$P''(t) = 9000\left[\frac{(2t^2+3)^2 - 2(2t^2+3)\cdot 4t \cdot t}{(2t^2+3)^4}\right]$$

$$= 9000\left[\frac{(2t^2+3) - 8t^2}{(2t^2+3)^3}\right]$$

$$= 9000\left[\frac{-6t^2+3}{(2t^2+3)^3}\right] = -27000\left[\frac{2t^2-1}{(2t^2+3)^3}\right]$$

$P''(t) = 0 \rightarrow 2t^2 - 1 = 0$

$t^2 = \frac{1}{2}$

$t = \sqrt{\frac{1}{2}}$ or $\cancel{t = -\sqrt{\frac{1}{2}}}$

$\approx .707$ months

Max growth rate?

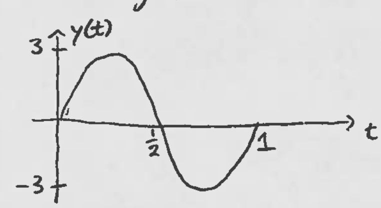
$P'\left(\sqrt{\frac{1}{2}}\right) = \dfrac{1125\sqrt{2}}{4} \approx 397.7$ squirrels per month

#22 $y(t) = 3\sin(2\pi t)$

(a) $v(t) = y'(t) = 6\pi\cos(2\pi t)$

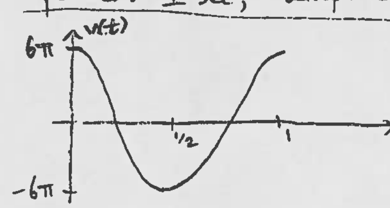
$a(t) = y''(t) = -12\pi^2 \sin(2\pi t)$

(b)

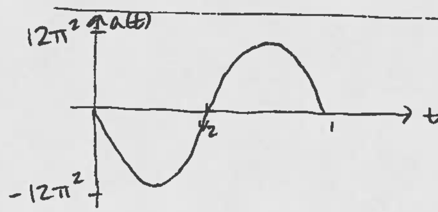


$y = a\sin(kt)$
period: $\frac{2\pi}{k}$
amp: a

period: 1 sec, amp: 3 in



period: 1 sec
amp: 6π in/sec



period: 1 sec
amp: $12\pi^2$ in/sec

(c) max/min position @ times $t = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots$

max position: 3 in. (amplitude)

when at max or min position, velocity $= 0$

(d) max/min velocity @ times $t = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

Max velocity: 6π in/sec (amplitude)

when at max/min velocity, acceleration $= 0$