MATH 1710.007, SPRING 2012

Exam 1 Review - Selected Solutions

#2

$s(1) = -16 + 128 = 112$ ft

$t = 1 + 2 = 3$ sec.

$s(3) = 240$ ft

$$v_{av} = \frac{s(3)-s(1)}{3-1} = \frac{240-112}{2} = 64 \text{ ft/sec}$$

$$\begin{aligned} s(1+h) &= -16(1+h)^2 + 128(1+h) \\ &= -16(1+2h+h^2) + 128 + 128h \\ &= -16h^2 - 32h - 16 + 128 + 128h \\ &= -16h^2 + 96h + 112 \end{aligned}$$

$$v_{av} = \frac{s(1+h) - s(1)}{h} = \frac{-16h^2 + 96h}{h} = -16h + 96$$

$$v_{inst} = \lim_{h\to 0} v_{av} = \lim_{h\to 0} -16h + 96 = 96 \text{ ft/sec}$$

#4

- $$\lim_{h\to 0} \frac{(7+h)^2 - 49}{h} = \lim_{h\to 0} \frac{49 + 14h + h^2 - 49}{h} = \lim_{h\to 0} \frac{h(14+h)}{h} = \lim_{h\to 0} 14 + h = \underline{\underline{14}}$$

top → 0
bottom → 0
$\frac{0}{0}$ type

- $$\lim_{h\to 0} \frac{\frac{1}{6+h} - \frac{1}{6}}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{6+h} - \frac{1}{6}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{6-(6+h)}{6(6+h)}\right] = \lim_{h\to 0} \frac{1}{h}\cdot\frac{-h}{6(6+h)} = \lim_{h\to 0} \frac{-1}{6(6+h)} = \underline{\underline{\frac{-1}{36}}}$$

top → 0
bottom → 0
$\frac{0}{0}$ type

- $$\lim_{x\to -2} \frac{x^2-4}{x^2+7x+10} = \lim_{x\to -2} \frac{(x+2)(x-2)}{(x+2)(x+5)} = \lim_{x\to -2} \frac{x-2}{x+5} = \underline{\underline{\frac{-4}{3}}}$$

top → 0
bottom → 0
"$\frac{0}{0}$" type

- $$\lim_{\theta\to\pi} \frac{\tan\theta}{\sin\theta} = \lim_{\theta\to\pi} \tan\theta \cdot \frac{1}{\sin\theta} = \lim_{\theta\to\pi} \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta} = \lim_{\theta\to\pi} \frac{1}{\cos\theta} = \frac{1}{\cos(\pi)} = \underline{\underline{-1}}$$

trig identity
$\tan\theta = \frac{\sin\theta}{\cos\theta}$

#5

- $$\lim_{h\to 0} \frac{\sqrt{64+h} - 8}{h} = \lim_{h\to 0} \frac{(\sqrt{64+h} - 8)}{(h)} \frac{(\sqrt{64+h} + 8)}{(\sqrt{64+h} + 8)} = \lim_{h\to 0} \frac{64 + h - 64}{h(\sqrt{64+h} + 8)} = \lim_{h\to 0} \frac{h}{h(\sqrt{64+h} + 8)} = \lim_{h\to 0} \frac{1}{\sqrt{64+h} + 8} = \frac{1}{\sqrt{64} + 8} = \underline{\underline{\frac{1}{16}}}$$

top → 0
bottom → 0
$\frac{0}{0}$ type

- $\lim_{\theta \to 0} \frac{1-\cos\theta}{\sin^2\theta}$

$= \lim_{\theta \to 0} \frac{(1-\cos\theta)}{(\sin^2\theta)} \cdot \frac{(1+\cos\theta)}{(1+\cos\theta)}$

$= \lim_{\theta \to 0} \frac{1-\cos^2\theta}{\sin^2\theta(1+\cos\theta)}$

Pythagorean identity
$\sin^2\theta + \cos^2\theta = 1$
So
$1-\cos^2\theta = \sin^2\theta$

$= \lim_{\theta \to 0} \frac{\sin^2\theta}{\sin^2\theta(1+\cos\theta)}$

$= \lim_{\theta \to 0} \frac{1}{1+\cos\theta} = \frac{1}{2}$

#6

$-1 \le \sin\left(\frac{1}{x}\right) \le 1$

and

$-x^4 \le x^4 \sin\left(\frac{1}{x}\right) \le x^4$.

$\lim_{x \to 0} -x^4 = 0$

↓ SQUEEZE

$\boxed{\lim_{x \to 0} x^4 \sin\left(\frac{1}{x}\right) = 0}$

↑ SQUEEZE

$\lim_{x \to 0} x^4 = 0$

#7

$\lim_{x \to \pi} 2|x-\pi| = 0$

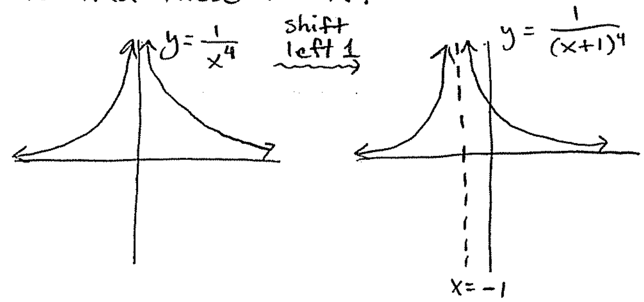
↓ SQUEEZE

$\boxed{\lim_{x \to \pi} f(x) = 0}$

↑ SQUEEZE

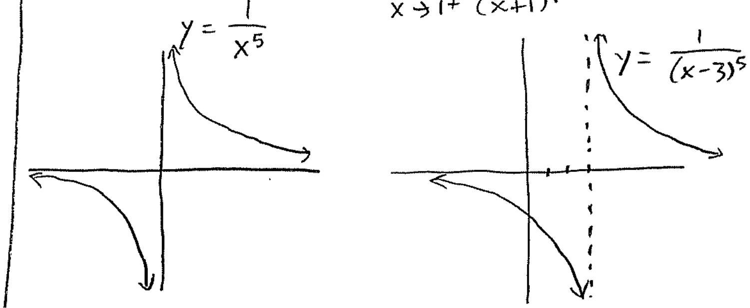
$\lim_{x \to \pi} -1-\cos x = -1-(-1) = 0$

#8 If you know how to do graph transformations, that's a quick way to find these limits.



$\lim_{x \to 1^-} \frac{1}{(x+1)^4} = +\infty$

$\lim_{x \to 1^+} \frac{1}{(x+1)^4} = +\infty$



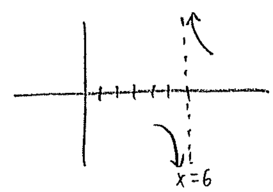
$\lim_{x \to 3^-} \frac{1}{(x-3)^5} = -\infty$

$\lim_{x \to 3^+} \frac{1}{(x-3)^5} = +\infty$

#9

- $f(x) = \frac{x^2+1}{x^2-5x-6} = \frac{x^2+1}{(x+1)(x-6)}$

top → 7
bottom → 0 $\frac{*}{0}$ type



Sign chart

sign of		6	
x^2+1	+		+
$(x+1)(x-6)$	(+)(−)		(+)(+)
$\frac{x^2+1}{(x+1)(x-6)}$	−		+

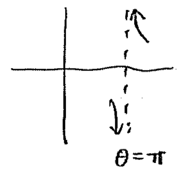
$\lim_{x \to 6^-} f(x) = -\infty$

$\lim_{x \to 6^+} f(x) = +\infty$

#9 cont

- $\lim\limits_{\theta \to \pi} \cot\theta = \lim\limits_{\theta \to \pi} \dfrac{\cos\theta}{\sin\theta}$

top $\to -1$
bottom $\to 0$
$\frac{*}{0}$ type



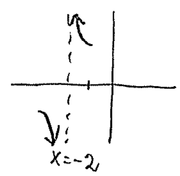
Sign chart

		π	
$\cos\theta$	$-$		$-$
$\sin\theta$	$+$		$-$
$\dfrac{\cos\theta}{\sin\theta}$	$-$		$+$

$\lim\limits_{\theta \to \pi^-} \cot\theta = -\infty$, $\lim\limits_{\theta \to \pi^+} \cot\theta = +\infty$

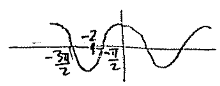
- $\lim\limits_{x \to -2} \dfrac{\cos x}{x^2 + 2x}$

top $\to \cos(-2)$
bottom $\to 0$
$\frac{*}{0}$ type



Sign chart

		-2	
$\cos x$	$-$		$-$
$x(x+2)$	$(-)(-)$		$(-)(+)$
$\dfrac{\cos x}{x(x+2)}$	$-$		$+$



$-\frac{\pi}{2} \approx -\frac{3}{2} = -1.5$

$-\frac{3\pi}{2} \approx -\frac{9}{2} = -4.5$

$\lim\limits_{x \to -2^-} \dfrac{\cos x}{x^2 + 2x} = -\infty$, $\lim\limits_{x \to -2^+} \dfrac{\cos x}{x^2 + 2x} = +\infty$

#12 $p(x) = -3x^{10} + 8x^3 - 4x^5 + x - 9$

leading term: $-3x^{10}$, basic shape ↙↘

$\lim\limits_{x \to \infty} p(x) = -\infty$, $\lim\limits_{x \to -\infty} p(x) = -\infty$

#13

- $\lim\limits_{x \to \pm\infty} \dfrac{3x^4 - 10x^2 + x - 6}{5x^4 - 13x + 11}$

$= \lim\limits_{x \to \pm\infty} \dfrac{x^4\left(3 - \frac{10}{x^2} + \frac{1}{x^3} - \frac{6}{x^4}\right)}{x^4\left(5 - \frac{13}{x^3} + \frac{11}{x^4}\right)}$

$= \lim\limits_{x \to \pm\infty} \dfrac{3 - \frac{10}{x^2} + \frac{1}{x^3} - \frac{6}{x^4}}{5 - \frac{13}{x^3} + \frac{11}{x^4}} = \dfrac{3}{5}$

$\boxed{\text{H.A. } y = \frac{3}{5}}$

- $\lim\limits_{x \to \pm\infty} \dfrac{2x^2 - 2x + 1}{6x^3 - 3x^2}$

$= \lim\limits_{x \to \pm\infty} \dfrac{x^2\left(2 - \frac{2}{x} + \frac{1}{x^2}\right)}{x^3\left(6 - \frac{3}{x}\right)}$

$= \lim\limits_{x \to \pm\infty} \dfrac{1}{x} \cdot \dfrac{\left(2 - \frac{2}{x} + \frac{1}{x^2}\right)}{\left(6 - \frac{3}{x}\right)}$

$= \lim\limits_{x \to \pm\infty} \dfrac{1}{3x} = \underline{\underline{0}}$

$\boxed{\text{H.A. } y = 0}$

- $f(x) = \dfrac{4x^3 + 1}{2x^3 + \sqrt{16x^6 + 5x^4 + 1}}$

when x is big $\sim \dfrac{4x^3 + 1}{2x^3 + \sqrt{16x^6}} = \dfrac{4x^3 + 1}{2x^3 + 4|x^3|}$

$\lim\limits_{x \to \infty} \dfrac{4x^3 + 1}{2x^3 + \sqrt{16x^6 + 5x^4 + 1}}$

$= \lim\limits_{x \to \infty} \dfrac{4x^3 + 1}{2x^3 + 4x^3}$ — $|x^3| = x^3$ if x is positive

$= \lim\limits_{x \to \infty} \dfrac{4x^3 + 1}{6x^3} = \dfrac{4}{6} = \dfrac{2}{3}$

$\lim\limits_{x \to -\infty} \dfrac{4x^3 + 1}{2x^3 + \sqrt{16x^6 + 5x^4 + 1}}$

$= \lim\limits_{x \to -\infty} \dfrac{4x^3 + 1}{2x^3 - 4x^3}$ — $|x^3| = -x^3$ if x is negative

$= \lim\limits_{x \to -\infty} \dfrac{4x^3 + 1}{-2x^3} = -\dfrac{4}{2} = -2$

$\boxed{\text{H.A. } y = \frac{2}{3} \text{ and } y = -2}$

#14

(a) $x^2-2x+1 \overline{\smash{)}\, x^4-2x^3-2x^2+6x-1}$, quotient x^2-3

$$\begin{array}{r} x^4-2x^3-2x^2+6x-1 \\ -x^4+2x^3-x^2 \\ \hline -3x^2+6x-1 \\ +3x^2-6x+3 \\ \hline 2 \end{array}$$

$$r(x) = x^2-3 + \frac{2}{x^2-2x+1}$$

$$Q(x) = x^2-3$$

(c) $r(x)$ has the same end behavior as $Q(x)$: ↑ ↗

$$\lim_{x\to\pm\infty} r(x) = +\infty$$

#15 $-1 \le \sin x \le 1$

$$4 \le \sin x + 5 \le 6$$

If x is positive,

$$\frac{4}{x} \le \frac{\sin x}{x} \le \frac{6}{x}$$

$$\lim_{x\to\infty} \frac{4}{x} = 0$$

↓ squeeze

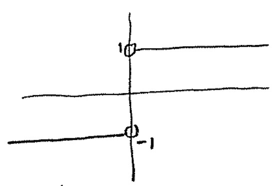
$$\boxed{\lim_{x\to\infty} \frac{\sin x}{x} = 0}$$

↑ squeeze

$$\lim_{x\to\infty} \frac{6}{x} = 0$$

#17

- $\frac{|x|}{x}$

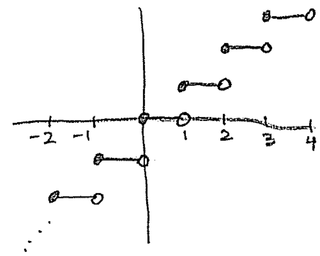


jump discontinuity at $x=0$

- $\sin(\frac{1}{x})$

oscillating discontinuity at $x=0$

- $\lfloor x \rfloor$ (floor function)



jump discontinuity at $x = \ldots, -2, -1, 0, 1, 2, \ldots$

- $R(x) = \frac{x^3-4x^2+4x}{x(x-1)}$

zeros of denominator? $x=0, 1$

$x \to 0$

top $\to 0$
bottom $\to 0$

$\frac{0}{0}$ type, hole in graph, removable discontinuity at $x=0$

$x \to 1$

top $\to 1$
bottom $\to 0$

$\frac{*}{0}$ type, vertical asymptote, infinite discontinuity at $x=1$

#18

$$f(x) = \begin{cases} a-3x^2 & x \le 2 \\ 5x-8 & x > 2 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} a-3x^2 = a-12$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 5x-8 = 2$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) \quad \text{if } \boxed{a=14}$$

Check $f(2) = \lim_{x\to 2} f(x)$: $f(2) = 14-3(4) = 2$.

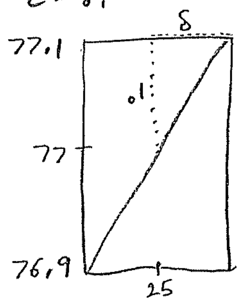
#19 $f(x) = x^4+3x-1$.

x	$f(x)$
-1	-3
1	3

The polynomial x^4+3x-1 is continuous. Since $f(-1)$ and $f(1)$ have opposite signs, the Intermediate Value Theorem guarantees there is a c between -1 and 1 with $f(c)=0$.

#21 $F(x) = \frac{9}{5}x + 32$

(a) $\varepsilon = .1$



$\frac{\text{rise}}{\text{run}} = \text{slope}$

$\frac{.1}{\delta} = \frac{9}{5}$

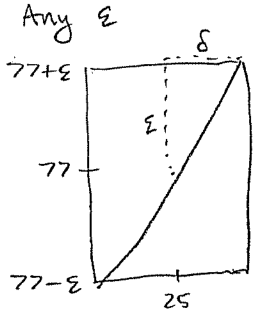
$9\delta = .5$

$\delta = \frac{.5}{9} = \frac{1}{18}$

biggest choice of δ: $\frac{1}{18} = .0\overline{5}$

the values .01, .001, .0001 would also work

(b) Any ε



$\frac{\text{rise}}{\text{run}} = \text{slope}$

$\frac{\varepsilon}{\delta} = \frac{9}{5}$

$\boxed{\delta = \frac{5\varepsilon}{9}}$

If $0 < |x-25| < \frac{5\varepsilon}{9}$,

then $|F(x) - 77| < \varepsilon$.

#22

- x^2, x^3 SEE NOTES FROM CLASS

- $f(x) = \frac{1}{x}$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{x+h} - \frac{1}{x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x-(x+h)}{x(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{x(x+h)}$$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

$$\boxed{\begin{aligned} y &= \frac{1}{x} \\ y' &= -\frac{1}{x^2} \end{aligned}}$$

- $f(x) = \sqrt{x}$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{(h)} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\boxed{\begin{aligned} y &= \sqrt{x} \\ y' &= \frac{1}{2\sqrt{x}} \end{aligned}}$$

- $f(x) = \frac{1}{\sqrt{x}}$

$$\lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right]$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{-1}{\sqrt{x}\sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}}$$

$$= \frac{1}{2\sqrt{x}} \cdot \frac{-1}{x} = \frac{-1}{2x^{1/2}x^{1}} = \frac{-1}{2x^{3/2}}$$

$$= -\frac{1}{2}x^{-3/2}$$

#23

$f(x) = x^2$

$f'(x) = 2x$

point: $(3, 9)$

slope: $f'(3) = 2(3) = 6$

tangent line:

$y = 6(x-3) + 9$

#24 $f(x) = |x|$