DRINFELD HECKE ALGEBRAS FOR SYMMETRIC GROUPS IN POSITIVE CHARACTERISTIC

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Abstract. We investigate deformations of skew group algebras arising from the action of the symmetric group on polynomial rings over fields of arbitrary characteristic. Over the real or complex numbers, Lusztig's graded affine Hecke algebra and analogs are all isomorphic to Drinfeld Hecke algebras, which include the symplectic reflection algebras and rational Cherednik algebras. Over fields of prime characteristic, new deformations arise that capture both a disruption of the group action and also a disruption of the commutativity relations defining the polynomial ring. We classify deformations for the symmetric group acting via its natural (reducible) reflection representation.

1. Introduction

Deformations of skew group algebras constructed from finite groups acting on polynomial rings are used in representation theory, combinatorics, and the study of orbifolds. These deformations include graded affine Hecke algebras, Drinfeld Hecke algebras, rational Cherednik algebras, and symplectic reflection algebras. The skew group algebra $S\#G$ arising from a group G acting on an algebra S by automorphisms is the natural semidirect product algebra $S \rtimes G$.

Lusztig [17, 18] defined deformations of skew group algebras $S\#G$ for a Weyl group G acting on its reflection representation $V = \mathbb{R}^n$ and its associated polynomial ring $S = S(V) = \mathbb{R}[v_1, \dots, v_n]$ in his investigations of the representation theory of groups of Lie type. His algebras alter the relations $g \cdot s = g(s) \cdot g$ capturing the group action but preserve the commutativity relation $v_i v_j = v_j v_i$ defining the polynomial ring S. These algebras are known as *graded affine Hecke algebras*. Around the same time, Drinfeld [10] more broadly considered an arbitrary subgroup G of $\mathrm{GL}_n(\mathbb{C})$. He defined a deformation of $S\#G$, sometimes called the *Drinfeld Hecke algebra*, by instead altering the commutation relation $v_i v_j = v_j v_i$ defining the polynomial ring S (now defined over $\mathbb{C}$) but leaving the group action relation alone. Drinfeld's type of deformation was rediscovered by Etingof and Ginzburg [11] in the study of orbifolds as *symplectic reflection algebras* when the acting group G is symplectic, which include *rational Cherednik algebras* as a special case. Collectively, we call all these deformations *Drinfeld Hecke algebras*.

Over $\mathbb{C}$, every deformation modeled on Lusztig's graded affine Hecke algebra is isomorphic to a deformation modeled on Drinfeld's Hecke algebra (see [19]). This result holds for any finite group G in the *nonmodular setting*, i.e., when the characteristic of the underlying field $\mathbb{F}$ is coprime to the group order $|G|$ (see [21]), and in the present work we extend this result

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to algebras that deform *both* Lusztig and Drinfeld type relations. The theorem fails in the *modular setting*, when char($\mathbb{F}$) divides $|G|$, and new types of deformations arise. We begin here a concrete study of these mixed deformations using PBW conditions from [21].

We consider the symmetric group $\mathfrak{S}_n$ acting by permutations of basis elements on a vector space $V \cong \mathbb{F}^n$ over a field $\mathbb{F}$. We include the case when char($\mathbb{F}$) divides the order of $\mathfrak{S}_n$. We deform the natural semi-direct product algebra $S \rtimes \mathfrak{S}_n$ for $S = S(V) \cong \mathbb{F}[v_1, \ldots, v_n]$, the polynomial ring, by introducing relations that disturb both the action of $\mathfrak{S}_n$ on V and the commutativity of the polynomial ring S. We classify the deformations arising in this way.

Recently, authors have considered the representation theory of similar deformations over fields of positive characteristic defined by relations of Drinfeld type. For example, Bezrukavnikov, Finkelberg, and Ginzburg [4], Devadas and Sun [9], Devadas and Sam [8], Cai and Kalinov [6], and Lian [16] all consider the Weyl group of type A_{n-1} (the irreducible reflection action of the symmetric group $\mathfrak{S}_n$) acting on $V \oplus V^*$ to give rational Cherednik algebras. Bellamy and Martino [3] as well as Gordon [13] investigate the action of the symmetric group $\mathfrak{S}_n$ in the nonmodular setting, when char($\mathbb{F}$) and $|\mathfrak{S}_n|$ are coprime. For related algebras built on the action of the special linear group, general linear group, and cyclic groups, see Balagović and Chen [1, 2] and Latour [14]. In contrast, the algebras classified here are defined by relations of both Lusztig and Drinfeld type.

Example 1.1. *Let $\mathbb{F}$ be a field of characteristic $p \geq 0$, $p \neq 2$, and let $G = \mathfrak{S}_n$ act on $V = \mathbb{F}^n$ by permuting basis vectors $v_1, \ldots, v_n$ for $n > 2$. Using Theorem 3.2 below, one may show that the $\mathbb{F}$-algebra generated by v in V and the group algebra $\mathbb{F}G$ with relations*

$$v_i v_j - v_j v_i = \sum_{k \neq i,j} \big((i\ j\ k) - (j\ i\ k) \big) \text{ in } \mathbb{F}G \quad \text{ and } \quad g v_i - \ g(v_i) g = \big(g(i) - i \big) g \text{ in } \mathbb{F}G$$

is a PBW deformation of $\mathbb{F}[v_1, v_2, ..., v_n] \rtimes G$.

Outline. We review Drinfeld Hecke algebras in Section 2 and PBW conditions in Section 3. In Section 4, we show that every Drinfeld Hecke algebra in the nonmodular setting is isomorphic to one in which the skew group relation is not deformed. We turn to the modular setting for the rest of the article and consider the symmetric group in its permutation representation. We examine PBW conditions in Sections 5 and 6. We classify the Drinfeld Hecke algebras in Section 7 and give these algebras explicitly for dimensions $n = 1, 2, 3$ in Section 8. We conclude by investigating the case of an invariant parameter in Section 9.

Notation. Throughout, we fix a field $\mathbb{F}$ of characteristic $p \geq 0$, $p \neq 2$, and unadorned tensor symbols denote the tensor product over $\mathbb{F}$: $\otimes = \otimes_{\mathbb{F}}$. We assume all $\mathbb{F}$-algebras are associative with unity $1_{\mathbb{F}}$ which is identified with the group identity 1_G in any group algebra $\mathbb{F}G$.

2. Drinfeld Hecke algebras

We consider a finite group $G \subset \mathrm{GL(V)}$ acting linearly on a finite-dimensional vector space $V \cong \mathbb{F}^n$ over the field $\mathbb{F}$. We may extend the action to one by automorphisms on the symmetric algebra $S(V) \cong \mathbb{F}[v_1, \ldots, v_n]$ for an $\mathbb{F}$-basis $v_1, \ldots, v_n$ of V and on the tensor algebra $T(V) = T_{\mathbb{F}}(V)$ (i.e., the free associative $\mathbb{F}$-algebra generated by $v_1, \ldots, v_n$). We write the action of G on any $\mathbb{F}$-algebra A with left superscripts, $a \mapsto {}^g a$ for g in G, a in A, to avoid confusion with the product $g\,a$ in any algebra containing $\mathbb{F}G$ and A.

Skew group algebra. Recall that the **skew group algebra** $R\#G = R \rtimes G$ of a group G acting by automorphisms on an $\mathbb{F}$-algebra R (e.g., $S(V)$ or $T(V)$) is the $\mathbb{F}$-algebra generated by R and $\mathbb{F}G$ with relations $g\,r = {}^g r\,g$ for r in R, g in G. This natural semi-direct product algebra is also called the *smash product* or *crossed product algebra*.

Relations given in terms of parameter functions. We consider an algebra generated by v in V and g in G with relations given in terms of two parameter functions λ and κ. The parameter λ measures the failure of group elements to merely act when they are interchanged with a vector v in V, and κ measures the failure of vectors in V to commute.

Let $\mathcal{H}_{\lambda,\kappa}$ be the associative $\mathbb{F}$-algebra generated by v in V together with the group algebra $\mathbb{F}G$ with additional relations

$$vw - wv = \kappa(v,w) \quad \text{and} \quad gv - {}^g v g = \lambda(g,v) \quad \text{for } \ v,w \in V, \ g \in G \tag{2.1}$$

for a choice of linear parameter functions λ and κ, with κ alternating,

$$\kappa : V \otimes V \to \mathbb{F}G, \quad \lambda : \mathbb{F}G \otimes V \to \mathbb{F}G\,.$$

For ease with notation, we write $\kappa(v,w)$ for $\kappa(v \otimes w)$ and $\lambda(g,v)$ for $\lambda(g \otimes v)$ throughout. Thus $\mathcal{H}_{\lambda,\kappa}$ is a quotient of the free $\mathbb{F}$-algebra generated by G and the basis $v_1, \ldots, v_n$ of V:

$$\begin{aligned}\mathcal{H}_{\lambda,\kappa} = \mathbb{F}\langle v_1, \ldots, v_n, g : g \in G\rangle / (gh - g \cdot_G h,& \\ vw - wv - \kappa(v,w),& \\ gv - {}^g v g - \lambda(g,v) : v,w \in V, \ g,h \in G)\,.&\end{aligned}$$

Note that $\mathcal{H}_{0,0} = S(V)\#G$, which is isomorphic to $S(V) \otimes \mathbb{F}G$ as an $\mathbb{F}$-vector space.

PBW property. Recall that a filtered algebra satisfies the PBW property with respect to a set of generating relations when its homogeneous version coincides with its associated graded algebra. We filter the algebra $\mathcal{H}_{\lambda,\kappa}$ by degree after assigning degree 1 to each v in V and degree 0 to each g in G. The homogeneous version of $\mathcal{H}_{\lambda,\kappa}$ is then the skew group algebra $S(V)\#G$, while the associated graded algebra $\operatorname{gr} \mathcal{H}_{\lambda,\kappa}$ is a quotient of $S(V)\#G$ (e.g., see Li [15, Theorem 3.2] or Braverman and Gaitsgory [5]). Thus we say $\mathcal{H}_{\lambda,\kappa}$ exhibits the *Poincaré-Birkhoff-Witt (PBW) property* when

$$\operatorname{gr} \mathcal{H}_{\lambda,\kappa} \cong S(V)\#G$$

as graded algebras. We call $\mathcal{H}_{\lambda,\kappa}$ a *Drinfeld Hecke algebra* in this case. Note that every Drinfeld Hecke algebra has $\mathbb{F}$-vector space basis $\{v_1^{i_1} v_2^{i_2} \cdots v_n^{i_n} g : i_m \in \mathbb{N}, g \in G\}$ for any $\mathbb{F}$-vector space basis $v_1, \ldots, v_n$ of V.

This terminology arises from the fact that Lusztig's graded versions of the affine Hecke algebras (see [17, 18]) are special cases of the PBW algebras $\mathcal{H}_{\lambda,\kappa}$: Lusztig's algebras arise in the case that $\kappa \equiv 0$ and G is a finite Coxeter group. (The parameter λ may be defined using a simple root system for G and BGG operators). Drinfeld's algebras [10] arise in the case that $\lambda \equiv 0$. Examples when $\lambda \equiv 0$ include rational Cherednik algebras and symplectic reflection algebras (see Etingof and Ginzburg [11]).

3. Poincaré-Birkhoff-Witt property

In this section, we recall necessary and sufficient conditions on the parameters λ and κ from [21] for a filtered quadratic algebra $\mathcal{H}_{\lambda,\kappa}$ to exhibit the PBW property. Again, let $G \subset \mathrm{GL}(V)$ be a finite group acting on $V \cong \mathbb{F}^n$. Let

$$\lambda_g : \mathbb{F}G \otimes V \to \mathbb{F} \quad \text{and} \quad \kappa_g : V \otimes V \to \mathbb{F}$$

be the coefficient functions for the parameters λ and κ, i.e., the $\mathbb{F}$-linear maps for which

$$\lambda(h,v) = \sum_{g\in G} \lambda_g(h,v) g \quad \text{and} \quad \kappa(u,v) = \sum_{g\in G} \kappa_g(u,v) g \quad \text{for } g,h \in G,\ u,v \in V.$$

Group action on parameters. The group G acts on the space of parameter functions λ and κ in the usual way,

$$({}^h\kappa)(v,w) = {}^h\big(\kappa({}^{h^{-1}}v,\ {}^{h^{-1}}w)\big),\ ({}^h\lambda)(g,v) = {}^h\big(\lambda(h^{-1}gh,\ {}^{h^{-1}}v)\big), \tag{3.1}$$

using the action of G on $\mathbb{F}G$ by conjugation, $g : h \mapsto ghg^{-1}$.

PBW conditions. The second PBW condition below measures the failure of κ to be G-invariant while the first shows that λ is determined by its values on generators of G.

Theorem 3.2 (PBW Conditions). *For any finite group $G \subset GL(V)$, an algebra $\mathcal{H}_{\lambda,\kappa}$ exhibits the PBW property if and only if the following five conditions hold for all g,h in G, u,v,w in V:*

(1) $\lambda(gh,v) = \lambda(g,{}^hv)h + g\lambda(h,v)$ *in* $\mathbb{F}G$.
(2) $\kappa({}^gu,{}^gv)g - g\kappa(u,v) = \lambda\big(\lambda(g,v),u\big) - \lambda\big(\lambda(g,u),v\big)$ *in* $\mathbb{F}G$.
(3) $\lambda_h(g,v)({}^hu - {}^gu) = \lambda_h(g,u)({}^hv - {}^gv)$ *in* V.
(4) $\kappa_g(u,v)({}^gw - w) + \kappa_g(v,w)({}^gu - u) + \kappa_g(w,u)({}^gv - v) = 0$ *in* V.
(5) $\lambda\big(\kappa(u,v),w\big) + \lambda\big(\kappa(v,w),u\big) + \lambda\big(\kappa(w,u),v\big) = 0$ *in* $\mathbb{F}G$.

We will refer to the specific conditions in this theorem, each with universal quantifiers, as *PBW Conditions (1) through (5)*. Notice that if $\mathcal{H}_{\lambda,\kappa}$ exhibits the PBW property, so does $\mathcal{H}_{c\lambda,c^2\kappa}$ for any c in $\mathbb{F}$.

We will need two corollaries from [21], the first on κ and the second on λ. We set $\operatorname{Ker}\kappa_g = \{v \in V : \kappa_g(v,w) = 0 \text{ for all } w \in V\}$ and say that κ is *supported* on a subset $S \subset G$ when $\kappa_h \equiv 0$ unless h lies in S. Similarly, we say that $\lambda(g,*)$ is *supported* on $S \subset G$ when $\lambda_h(g,v) = 0$ for all v in V and h not in S. Here, $V^g = \{v \in V : {}^gv = v\}$ is the fixed point space.

Corollary 3.3. *Let $G \subset GL(V)$ be a finite group and assume $\mathcal{H}_{\lambda,\kappa}$ exhibits the PBW property. For every g in G, if $\kappa_g \not\equiv 0$, then either*

(a) $\operatorname{codim} V^g = 0$ *(i.e., g acts as the identity on V), or*
(b) $\operatorname{codim} V^g = 1$ *and $\kappa_g(v_1,v_2) = 0$ for all v_1,v_2 in V^g, or*
(c) $\operatorname{codim} V^g = 2$ *with* $\operatorname{Ker}\kappa_g = V^g$.

Recall that a nonidentity element of $\mathrm{GL}(V)$ is a *reflection* if it fixes a hyperplane of V pointwise. We see in the above corollary that over fields of positive characteristic, a parameter κ defining an algebra with the PBW property may be supported on reflections in addition to the identity 1_G and bireflections (elements whose fixed point space has codimension 2) in contrast to the possible support in the nonmodular setting.

Corollary 3.4. *Let $G \subset GL(V)$ be a finite group and say $\mathcal{H}_{\lambda,\kappa}$ exhibits the PBW property. Then the following statements hold for any g in G.*

(1) *$\lambda(1, *)$ is identically zero: $\lambda(1, v) = 0$ for all v in V.*
(2) *$\lambda(g, *)$ determines $\lambda(g^{-1}, *)$ by $g\lambda(g^{-1}, v) = -\lambda(g, {}^{g^{-1}}v)g^{-1}$.*
(3) *$\lambda(g, *)$ can be defined recursively: For $j \geq 1$, $\lambda(g^j, v) = \sum_{i=0}^{j-1} g^{j-1-i}\, \lambda(g, {}^{g^i}v)\, g^i$.*
(4) *$\lambda(g, *)$ is supported on h in G with $h^{-1}g$ either a reflection or the identity on V. If $h^{-1}g$ is a reflection, then $\lambda_h(g, v) = 0$ for all v on the reflecting hyperplane $V^{h^{-1}g}$.*
(5) *If $V^g \neq V$, $\lambda_1(g, v) = 0$ unless g is a reflection and $v \notin V^g$.*

We give an example in which the parameter $\kappa : V \otimes V \to \mathbb{F}G$ is *not* G-invariant.

Example 3.5. Let $G = \mathfrak{S}_n$ act on $V = \mathbb{F}^n$ for $n > 3$ by permuting basis elements $v_1, \ldots, v_n$, i.e., $v_i \mapsto {}^g v_i = v_{g(i)}$ for g in G. Fix two scalar parameters m, m' in $\mathbb{F}$. Define a_{ij} in $\mathbb{F}$ for $1 \leq i \neq j \leq n$ by $m = a_{12} = a_{13} = -a_{21} = -a_{31}$, $m' = a_{23} = -a_{32}$, and $a_{ij} = 0$ otherwise. Then the algebra defined by

$$v_1v_2 - v_2v_1 = v_2v_3 - v_3v_2 = v_3v_1 - v_1v_3 = m^2\big((1\ 3\ 2) - (1\ 2\ 3)\big),\ v_iv_j - v_jv_i = 0 \text{ otherwise},$$

$$gv_i - v_{g(i)}g = \sum_{j \neq i} (a_{ij} - a_{g(i)\,g(j)})\ g(i\ j) \quad \text{for } g \in G,\ 1 \leq i \leq n$$

is a PBW deformation of $S(V)\#\mathbb{F}G$ (see Theorem 7.4 with $c = a_{123}$). Notice (see Eq. (5.1) and Definition 7.1) that $a_{ij} = (1/4)\lambda_1\big((i\ j), v_i - v_j\big)$.

4. Nonmodular Setting

Before classifying algebras in the modular setting, we verify in this section that every Drinfeld Hecke algebra $\mathcal{H}_{\lambda,\kappa}$ in the nonmodular setting is isomorphic to one with $\lambda \equiv 0$. We consider an arbitrary finite group $G \subset \mathrm{GL}(V)$ acting on $V \cong \mathbb{F}^n$ but assume $\mathrm{char}(\mathbb{F}) \neq 2$ does not divide $|G|$ (e.g., $\mathrm{char}(\mathbb{F}) = 0$) in this section only. In the next theorem, we extend a result of [21] from the special case in which one of the parameter functions is zero to the case of more general parameters; the result of [21] strengthened a theorem in [19] from the setting of Coxeter groups to arbitrary finite groups.

Theorem 4.1. *Say $G \subset GL(V)$ is a finite group for $V \cong \mathbb{F}^n$ with $char(\mathbb{F}) \neq 2$ coprime to $|G|$. If an algebra $\mathcal{H}_{\lambda,\kappa'}$ exhibits the PBW property for parameters $\lambda : \mathbb{F}G \otimes V \to \mathbb{F}G$ and $\kappa' : V \otimes V \to \mathbb{F}G$, then there is an algebra $\mathcal{H}_{0,\kappa}$ for some parameter $\kappa : V \otimes V \to \mathbb{F}G$ with*

$$\mathcal{H}_{0,\kappa} \ \cong\ \mathcal{H}_{\lambda,\kappa'} \qquad \textit{as filtered algebras.}$$

Proof. Define a conversion function $\gamma : V \to \mathbb{F}G$ by

$$\gamma(v) = \frac{1}{|G|} \sum_{a,b \in G} \lambda_{ab}(b, {}^{b^{-1}}v)a \quad \text{and write } \gamma = \sum_{a \in G} \gamma_a a$$

for coefficient functions $\gamma_a : V \to \mathbb{F}$. Let $\kappa : V \otimes V \to \mathbb{F}G$ be the parameter function

$$\kappa(u, v) = \gamma(u)\gamma(v) - \gamma(v)\gamma(u) + \lambda(\gamma(u), v) - \lambda(\gamma(v), u) + \kappa'(u, v).$$

Consider the associative $\mathbb{F}$-algebra F generated by V and the algebra $\mathbb{F}G$, i.e.,

$$F = T_{\mathbb{F}}(\mathbb{F}G \oplus V)/(g \otimes h - gh, 1_{\mathbb{F}} - 1_G : g, h \in G)$$

(identifying each g with $(g, 0)$). Define an algebra homomorphism $f : F \to \mathcal{H}_{\lambda,\kappa'}$ by

$$f(v) = v + \gamma(v) \quad \text{and} \quad f(g) = g \quad \text{ for all } v \in V \text{ and } g \in G\,.$$

One may use the PBW conditions for $\mathcal{H}_{\lambda,\kappa'}$ to show that the relations defining $\mathcal{H}_{0,\kappa}$ lie in the kernel of f, as in the proof of [21, Theorem 4.1]. Thus f factors through an onto, filtered algebra homomorphism

$$f : \mathcal{H}_{0,\kappa} \twoheadrightarrow \mathcal{H}_{\lambda,\kappa'}\,.$$

The m-th filtered components of $\mathcal{H}_{\lambda,\kappa'}$ and $\mathcal{H}_{0,\kappa}$ are both spanned over $\mathbb{F}$ by the monomials $v_1^{a_1} \cdots v_n^{a_n} g$ for $g \in G$ and $a_i \in \mathbb{N}$ with $\sum_i a_i \leq m$, for a fixed basis $v_1, \dots, v_n$ of V. This spanning set is in fact a basis for $(\mathcal{H}_{\lambda,\kappa'})_m$ by the PBW property, and thus

$$\dim_{\mathbb{F}}(\mathcal{H}_{0,\kappa})_m \leq \dim_{\mathbb{F}}(\mathcal{H}_{\lambda,\kappa'})_m\,.$$

The map f restricts to a surjective linear transformation of finite-dimensional $\mathbb{F}$-vector spaces on each filtered piece,

$$f : (\mathcal{H}_{0,\kappa})_m \twoheadrightarrow (\mathcal{H}_{\lambda,\kappa'})_m\,,$$

and hence is injective on each filtered piece. (Indeed, for any $v_1^{a_1} \cdots v_n^{a_n} g$ of filtered degree m in the PBW basis for $\mathcal{H}_{\lambda,\kappa'}$, the element $(v_1 - \gamma(v_1))^{a_1} \cdots (v_n - \gamma(v_n))^{a_n} g$ in $\mathcal{H}_{0,\kappa}$ is a preimage under f and also has filtered degree m.) Thus f is an isomorphism of filtered algebras. Notice that f in turn induces an isomorphism of graded algebras, $\operatorname{gr} \mathcal{H}_{0,\kappa} \cong \operatorname{gr} \mathcal{H}_{\lambda,\kappa'} \cong S(V)\#G$, and $\mathcal{H}_{0,\kappa}$ also exhibits the PBW property. □

The special case of Theorem 4.1 when $\kappa' \equiv 0$ is from [21]. Note that Theorem 4.1 fails over fields of positive characteristic as the next example from [21] shows: not every algebra $\mathcal{H}_{\lambda,0}$ (modeled on Lusztig's graded affine Hecke algebra) is isomorphic to an algebra $\mathcal{H}_{0,\kappa}$ (modeled on the Drinfeld Hecke algebra).

Example 4.2. Let $G \cong \mathbb{Z}/2\mathbb{Z}$ be generated by $g = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ acting on $V = \mathbb{F}_2^2$ with respect to an ordered basis v, w. Consider the $\mathbb{F}$-algebra $\mathcal{H}_{\lambda,\kappa'}$ generated by V and $\mathbb{F}G$ with relations

$$gv = vg, \quad gw = vg + wg + 1, \quad vw - wv = g\,.$$

Then $\mathcal{H}_{\lambda,\kappa'}$ satisfies the PBW property but is not isomorphic to $\mathcal{H}_{0,\kappa}$ for any parameter κ. (Here, $\lambda(g, v) = \lambda(1, v) = \lambda(1, w) = 0$, $\lambda(g, w) = 1$, and $\kappa'(u, v) = g$.)

Theorem 4.1 and PBW Condition (2) (with $\lambda \equiv 0$) imply the next observation.

Corollary 4.3. *Every Drinfeld Hecke algebra in the nonmodular setting arises from a parameter κ that is* invariant*, i.e., satisfying*

$$\kappa({}^g u, {}^g v)g = g\kappa(u, v) \quad \textit{ for } g \in G,\ u, v \in V.$$

5. Deforming the group action

In this section and the next, we lay the framework for a complete classification of Drinfeld Hecke algebras for the symmetric group $\mathfrak{S}_n$ acting by permutation matrices on $V \cong \mathbb{F}^n$. In Sections 7 and 8, we classify these algebras by giving the parameters λ and κ such that $\mathcal{H}_{\lambda,\kappa}$ is PBW. In this section we obtain the form of the parameter λ, and in the next section we describe the parameter κ. We assume $n > 2$ here and in Sections 6 and 7 for ease with notation; we give the PBW algebras explicitly for $n = 1, 2, 3$ in Section 8.

We consider the action of the symmetric group by permutations. Let $G = \mathfrak{S}_n$ act on $V \cong \mathbb{F}^n$ by permuting basis elements $v_1, \dots, v_n$ of V, i.e., $v_i \mapsto {}^g v_i = v_{g(i)}$ for g in G. We write $s_i = (i\ \ i+1)$ for the adjacent transpositions generating G for $1 \leq i < n$ and set $s_n = (n\ 1)$ for ease with later notation. Recall that we assume $2 \neq \mathrm{char}(\mathbb{F}) \geq 0$. Fix linear parameter functions

$$\kappa : V \otimes V \to \mathbb{F}G \quad \text{and} \quad \lambda : \mathbb{F}G \otimes V \to \mathbb{F}G, \quad \text{with } \kappa \text{ alternating.}$$

Scalar parameters of freedom. We show how each PBW algebra $\mathcal{H}_{\lambda,\kappa}$ has parameter λ determined by certain values on reflections. Note the group action in Eq. (3.1) induces the usual action on $\lambda_1 : G \otimes V \to \mathbb{F}$ with

$$({}^{h^{-1}}\lambda_1)\big((i\ j), v_i\big) = \lambda_1\big(h(i\ j)h^{-1},\ {}^h v_i\big) = \lambda_1\big((h(i), h(j)), v_{h(i)}\big)\ .$$

We define scalars in $\mathbb{F}$ for any linear parameter function $\lambda : \mathbb{F}G \otimes V \to \mathbb{F}G$:

$$\begin{aligned} \alpha_{ij} &:= \tfrac{1}{4}\ \lambda_1\big((i\ j), v_i - v_j\big), \\ \alpha_{ijk} &:= \alpha_{ij}\alpha_{jk} + \alpha_{jk}\alpha_{ki} + \alpha_{ki}\alpha_{ij}, \\ {}^g\alpha_{ij} &:= \tfrac{1}{4}\ \lambda_1\big((g(i)\ g(j)), v_{g(i)} - v_{g(j)}\big), \\ {}^g\alpha_{ijk} &:= \alpha_{g(i)g(j)}\alpha_{g(j)g(k)} + \alpha_{g(j)g(k)}\alpha_{g(k)g(i)} + \alpha_{g(k)g(i)}\alpha_{g(i)g(j)}, \text{ and} \\ \beta_k &:= \tfrac{1}{2}\ \lambda_{s_k}\big(s_k, v_k - v_{k+1}\big), \end{aligned} \tag{5.1}$$

for $1 \leq i, j, k \leq n$ with i, j, k distinct. We take indices on β modulo n throughout to more easily cyclically permute parameters in later results. We will see that if $\mathcal{H}_{\lambda,\kappa}$ satisfies the PBW property, then $\beta_k = \lambda_{s_k}(s_k, v_k)$ for all k (by Lemma 5.5).

Determination of λ. We prove in this section that every PBW algebra $\mathcal{H}_{\lambda,\kappa}$ has parameter λ determined by its values α_{ij} and β_k:

Theorem 5.2. *A parameter λ satisfies PBW Conditions (1) and (3) if and only if*

$$\lambda(g, v_i) \ = \sum_{k=0}^{g(i)-i+n-1} \beta_{i+k}\, g + \sum_{1\ \leq\ j \neq i\ \leq\ n} (\alpha_{ij} -\ {}^g\alpha_{ij})\, g(i\ j) \tag{5.3}$$

for all $g \in G$ and $1 \leq i \leq n$ with $\beta_1 + \cdots + \beta_n = 0$.

We collect some necessary observations before giving the proof of this theorem at the end of this section.

Lemmas for the proof of Theorem 5.2. Recall the *(absolute) reflection length* function $\ell : \mathfrak{S}_n \to \mathbb{Z}_{\geq 0}$ on $\mathfrak{S}_n$ which gives the minimal number $\ell(g)$ of transpositions in a factorization of g into transpositions. The following observation is well-known for reflection groups over $\mathbb{R}$ (e.g., see [7], [12], [20]) and we include a proof for the symmetric group acting over arbitrary fields for the sake of completeness.

Lemma 5.4. *For g, h in $\mathfrak{S}_n$, $\ell(g) = \mathrm{codim}\, V^g$, and $V^g \cap V^h = V^{gh}$ when $\ell(g) + \ell(h) = \ell(gh)$.*

Proof. For $\mathfrak{S}_n$ acting on $V_{\mathbb{R}} = \mathbb{R}^n$ by permutation of basis vectors, $\ell(g) = \mathrm{codim}\, V_{\mathbb{R}}^g$. But $\mathrm{codim}\, V_{\mathbb{R}}^g = \mathrm{codim}\, V^g$ (just consider the decomposition of g into disjoint cycles and take orbit sums for invariant vectors, for example). Hence $\ell(g) + \ell(h) = \ell(gh)$ if and only if $\mathrm{codim}\, V^g + \mathrm{codim}\, V^h = \mathrm{codim}\, V^{gh}$. In this case, $V^g \cap V^h = V^{gh}$ since $V^g \cap V^h \subset V^{gh}$ implies that $\mathrm{codim}\, V^g + \mathrm{codim}\, V^h - \mathrm{codim}(V^{gh}) \geq \mathrm{codim}\, V^g + \mathrm{codim}\, V^h - \mathrm{codim}(V^g \cap V^h) \geq 0.$ $\square$

Our next observation follows from PBW Condition (3) with $h = g(i\ j)$, $u = v_i, v = v_j$:

Lemma 5.5. *If the parameter function λ satisfies PBW Condition (3), then*

$$\lambda_{g(i\ j)}(g, v_i) = -\lambda_{g(i\ j)}(g, v_j) = \tfrac{1}{2}\lambda_{g(i\ j)}(g, v_i - v_j) \quad \text{for } i \neq j,\ g \in G.$$

In particular, $\lambda_1\big((i\ j), v_i\big) = -\lambda_1\big((i\ j), v_j\big) = \frac{1}{2}\lambda_1\big((i\ j), v_i - v_j\big)$.

Lemma 5.6. *If the parameter function λ satisfies PBW Condition (1), then $\lambda_c(c, v) = 0$ for all c in G and v in V^c, the fixed point space.*

Proof. We induct on the (absolute) reflection length $\ell(c)$ of c using Corollary 3.4 (1), which follows directly from PBW Condition (1) (see the proof of [21, Cor. 3.3]). First suppose c is a reflection itself with $v \in V^c$. Then by PBW Condition (1),

$$0 = \lambda(1, v) = \lambda(cc, v) = \lambda(c,\ {}^c v)c + c\lambda(c, v) = \lambda(c, v)c + c\lambda(c, v).$$

The coefficient of the identity group element on the right-hand side is $0 = 2\lambda_c(c, v)$.

Now suppose the claim holds for all group elements g with $\ell(g) = k$ and that $\ell(c) = k+1$. Then $c = ab$ for some a with $\ell(a) = k$ and some transposition b. As $\ell(ab) = \ell(a) + \ell(b)$, the vector v lies in $V^{ab} = V^a \cap V^b$ by Lemma 5.4. By PBW Condition (1), $\lambda(c, v) = \lambda(a, {}^b v)b + a\lambda(b, v)$, and the result follows from the induction hypothesis applied to the terms with c:

$$\lambda_c(c, v) = \lambda_a(a, v) + \lambda_b(b, v) = 0.$$

□

Lemma 5.7. *Say the parameter function λ satisfies PBW Condition (1). Then*

(a) *For any $g \in G$ and any i, $\lambda_g(g, v_i) = \lambda_{(i\ g(i))}\big((i\ g(i)), v_i\big) = -\lambda_{(i\ g(i))}\big((i\ g(i)), v_{g(i)}\big)$.*

(b) *For any k-cycle $(l_1\ l_2\ \cdots\ l_k)$ in G,*

$$\lambda_{(l_1\ l_2)}\big((l_1\ l_2), v_{l_1}\big) + \lambda_{(l_2\ l_3)}\big((l_2\ l_3), v_{l_2}\big) + \cdots + \lambda_{(l_k\ l_1)}\big((l_k\ l_1), v_{l_k}\big) = 0\,.$$

(c) *For any $i < j$,*

$$\lambda_{(i\ j)}\big((i\ j), v_i\big) = \lambda_{s_i}\big(s_i, v_i\big) + \lambda_{s_{i+1}}\big(s_{i+1}, v_{i+1}\big) + \cdots + \lambda_{s_{j-1}}\big(s_{j-1}, v_{j-1}\big)\ .$$

Proof. For (a), assume that $v_i \notin V^g$, else the claim follows from Lemma 5.6. Set $j = g(i)$ and $h = g(i\ j)$. Then $v_j \in V^h$ and PBW Condition (1) implies that

$$\lambda(g, v_i) = \lambda(h, {}^{(i\ j)} v_i)(i\ j) + h\lambda\big((i\ j), v_i\big) = \lambda(h, v_j)(i\ j) + h\lambda\big((i\ j), v_i\big).$$

We isolate the coefficient of g and apply Lemma 5.6 twice:

$$\lambda_g(g, v_i) = \lambda_h(h, v_j) + \lambda_{(i\ j)}\big((i\ j), v_i\big) = \lambda_{(i\ j)}\big((i\ j), v_i\big) = -\lambda_{(i\ j)}\big((i\ j), v_j\big).$$

For (b), Lemma 5.6 and (a) imply that, for $g = (l_1\ \cdots\ l_k)$,

$$0 = \lambda_g(g, \sum_{i=1}^{k} v_{l_i}) = \sum_{i=1}^{k} \lambda_g(g, v_{l_i}) = \sum_{i=1}^{k-1} \lambda_{(l_i\ l_{i+1})}\big((l_i\ l_{i+1}), v_{l_i}\big) + \lambda_{(l_k\ l_1)}\big((l_k\ l_1), v_k\big).$$

For (c), just use (b) with cycle $(i\ i+1\ \cdots\ j)$ and (a). □

Remark 5.8. Note that Lemma 5.7 implies that there are at most $n-1$ choices determining the values $\lambda_g(g, v)$ for g in G and v in V in a PBW algebra $\mathcal{H}_{\lambda,\kappa}$. Indeed, the values of $\lambda_g(g, v)$ are determined by the values $\lambda_{(i\ j)}((i\ j), v_i)$ for $i < j$ by part (a), which are determined by the values $\beta_k = \lambda_{s_k}(s_k, v_k)$ for $1 \leq k < n$ by part (c).

Lemma 5.9. *If the parameter function λ satisfies PBW Conditions (1) and (3), then for all $g \in G$, $\lambda(g, v_1 + \cdots + v_n) = 0$.*

Proof. PBW Condition (3) implies Corollary 3.4 (4) (see the proof of [21, Cor. 3.3]), hence

$$\begin{aligned}\sum_{i=1}^{n} \lambda(g, v_i) &= \sum_{i=1}^{n} \lambda_g(g, v_i) g + \sum_{i=1}^{n} \sum_{j \neq i} \lambda_{g(i\ j)}(g, v_i) g(i\ j) \\ &= \lambda_g\big(g, \sum_{i=1}^{n} v_i\big) g + \sum_{1 \leq i < j \leq n} \big(\lambda_{g(i\ j)}(g, v_i) + \lambda_{g(i\ j)}(g, v_j)\big) g(i\ j).\end{aligned}$$

This vanishes by Lemma 5.5 and Lemma 5.6. $\square$

Lemma 5.10. *For the parameter function λ satisfying PBW Conditions (1) and (3),*

$$\beta_1 + \cdots + \beta_n = 0 \quad \text{and} \quad \lambda_g(g, v_i) = \sum_{k=0}^{g(i)-i+n-1} \beta_{i+k} \quad \text{for all } g \in G,\ 1 \leq i \leq n\,. \tag{5.11}$$

Proof. Lemma 5.7 with the cycle $(1\ 2\ \cdots\ n)$ implies that $\sum_{1 \leq j \leq n} \beta_j = 0$. For $i < g(i)$,

$$\sum_{k=0}^{g(i)-i+n-1} \beta_{i+k} = \sum_{k=i}^{g(i)-1} \beta_k + \sum_{k=g(i)}^{g(i)+n-1} \beta_k = \sum_{k=i}^{g(i)-1} \lambda_{s_k}(s_k, v_k),$$

which is just $\lambda_{(i\ g(i))}\big((i\ g(i)), v_i\big) = \lambda_g(g, v_i)$ by Lemma 5.7. Similarly, for $g(i) < i$,

$$\sum_{k=0}^{g(i)-i+n-1} \beta_{i+k} = \beta_i + \beta_{i+1} + \cdots + \beta_n + \beta_1 + \cdots + \beta_{g(i)-1} = -(\beta_{g(i)} + \cdots + \beta_{i-1}),$$

which again is just $-\lambda_{(g(i)\ i)}\big((g(i)\ i), v_{g(i)}\big) = \lambda_g(g, v_i)$ by Lemma 5.7. $\square$

Corollary 5.12. *If the parameter λ satisfies (5.11), then for any k-cycle $(l_1\ \cdots\ l_k)$ in G,*

$$\lambda_{(l_1\ l_2)}\big((l_1\ l_2), v_{l_1}\big) + \lambda_{(l_2\ l_3)}\big((l_2\ l_3), v_2\big) + \cdots + \lambda_{(l_k\ l_1)}\big((l_k\ l_1), v_{l_k}\big) = 0\,.$$

Proof. Since (5.11) implies that

$$\lambda_{(i\ j)}((i\ j), v_i) = \beta_i + \cdots + \beta_n + \beta_1 + \cdots + \beta_{j-1} \quad \text{for } 1 \leq i \neq j \leq n$$

(note that the sum stops at β_n when $j = 1$ as $\beta_n = \beta_0$), the sum $\sum_{i=1}^{k} \lambda_{(l_i\ l_{i+1})}((l_i\ l_{i+1}), v_{l_i})$ is a multiple of $\beta_1 + \cdots + \beta_n$, which is zero by the first equality in (5.11). $\square$

5.1. **Proof of Theorem 5.2.** We now have the tools to show that every PBW algebra $\mathcal{H}_{\lambda,\kappa}$ has parameter λ determined by the values α_{ij} and β_k.

Proof of Theorem 5.2. We show PBW Conditions (1) and (3) are equivalent to these three facts for all g in G and all i:

(a) $\lambda_{g(i\ j)}(g, v_i) = \alpha_{ij} - {}^{g}\alpha_{ij}$ for all $j \neq i$,
(b) $\beta_1 + \cdots + \beta_n = 0$ and $\lambda_g(g, v_i) = \sum_{k=0}^{g(i)-i+n-1} \beta_k$, and
(c) $\lambda(g, v_i)$ is supported on g and all $g(i\ j)$ for $j \neq i$.

Assume PBW Conditions (1) and (3) both hold. Lemma 5.10 implies (b). PBW Condition (3) implies part (4) of Corollary 3.4 (see the proof of [21, Cor. 3.3]), implying (c).

We induct on the (absolute) reflection length $\ell(g)$ of g in $G = \mathfrak{S}_n$ to verify (a). First, if $g = 1$, then both sides of (a) vanish by Corollary 3.4(1). Now suppose g is a transposition fixing i and j. Then $\alpha_{ij} = {}^g\alpha_{ij}$ and the right-hand side of (a) is zero; on the other hand, PBW Condition (1) implies that

$$\lambda\big((i\ j), v_i\big)g + (i\ j)\lambda(g, v_i) = \lambda(g, v_j)(i\ j) + g\lambda\big((i\ j), v_i\big)$$

and we apply Lemma 5.5 to the coefficient of g,

$$\lambda_1\big((i\ j), v_i\big) + \lambda_{g(i\ j)}(g, v_i) = \lambda_{g(i\ j)}(g, v_j) + \lambda_1\big((i\ j), v_i\big),$$

to see the left-hand side of (a) is zero. Now suppose instead $g = (i\ k)$ for some $k \neq i$. Then $g = (i\ j\ k)(i\ j) = (j\ k)(i\ j\ k)$, and we use PBW Condition (1) to write $\lambda((i\ k), v_i - v_j)$ in two ways and match the coefficients of $(i\ k)(i\ j)$:

$$\begin{aligned}
&\lambda_{(i\ k)(i\ j)}\big((i\ k), v_i - v_j\big) = \lambda_{(i\ j\ k)(i\ j)}\big((i\ j\ k), v_j - v_i\big) + \lambda_1((i\ j), v_i - v_j) \quad \text{ and}\\
&\lambda_{(i\ k)(i\ j)}\big((i\ k), v_i - v_j\big) = \lambda_1\big((j\ k), v_j - v_k\big) + \lambda_{(i\ j\ k)(i\ j)}\big((i\ j\ k), v_i - v_j\big);
\end{aligned}$$

we conclude $2\lambda_{(i\ k)(i\ j)}((i\ k), v_i - v_j) = 4(\alpha_{ij} - {}^g\alpha_{ij})$ and Lemma 5.5 implies (a).

To show the induction step, fix some g in G, and assume the result holds for all group elements with smaller (absolute) reflection length. Write $g = g_1g_2$ for some g_1, g_2 in $\mathfrak{S}_n$ with $0 < \ell(g_1), \ell(g_2) < \ell(g)$. PBW Condition (1) implies that

$$\lambda(g, v_i - v_j) = \lambda(g_1g_2, v_i - v_j) = \lambda\big(g_1, {}^{g_2}(v_i - v_j)\big)g_2 + g_1\lambda(g_2, v_i - v_j),$$

and we equate the coefficients of $g(i\ j) = g_1\big(g_2(i)\ g_2(j)\big)g_2$ to obtain (a):

$$\begin{aligned}
2\lambda_{g(i\ j)}(g, v_i) = \lambda_{g(i\ j)}(g, v_i - v_j) &= \lambda_{g_1(g_2(i)\ g_2(j))}(g_1, v_{g_2(i)} - v_{g_2(j)}) + \lambda_{g_2(i\ j)}(g_2, v_i - v_j)\\
&= 2({}^{g_2}\alpha_{ij} - {}^{g_1g_2}\alpha_{ij}) + 2(\alpha_{ij} - {}^{g_2}\alpha_{ij}) = 2(\alpha_{ij} - {}^g\alpha_{ij})\,.
\end{aligned}$$

To prove the converse, we assume (a), (b), and (c) hold and first show PBW Condition (1). Note that (b) implies that for all $g \in G$ and for all $v_i \in V$, $\lambda_g(g, v_i)$ coincides with $\lambda_{(i\ g(i))}((i\ g(i)), v_i)$. The right-hand side of PBW Condition (1) at $v = v_i$ is

$$\begin{aligned}
&\lambda_{(h(i)\ gh(i))}\big((h(i)\ gh(i)), v_{h(i)}\big)\, gh \ + \sum_{j:\, h(j)\neq h(i)} (\alpha_{h(i)h(j)} - {}^g\alpha_{h(i)h(j)})\, gh(i\ j)h^{-1}h\\
&\quad + \lambda_{(i\ h(i))}\big((i\ h(i)), v_i\big)\, gh \ + \sum_{j:j\neq i} (\alpha_{ij} - {}^h\alpha_{ij})\, gh(i\ j).
\end{aligned} \tag{5.13}$$

We apply Corollary 5.12 to the 3-cycle $(i\ h(i)\ gh(i))$ and the 2-cycle $(i\ gh(i))$, as well as (b), to simplify the gh terms and obtain

$$-\lambda_{(i\ gh(i))}\big((i\ gh(i)), v_{gh(i)}\big)\, gh = \lambda_{(gh(i)\ i)}\big((gh(i)\ i), v_i\big)\, gh \ = \lambda_{gh}\big(gh, v_i\big)\, gh\,.$$

The $gh(i\,j)$ terms combine as $\sum_{j:i\neq j} \lambda_{gh(i\ j)}(gh, v_i)\, gh(i\ j)$. Thus Eq. (5.13) is just $\lambda(gh, v_i)$.

The only nontrivial case in showing PBW Condition (3) is when $h = g(i\ j)$ with vectors $\{u, v\} = \{v_i, v_j\}$. To confirm that

$$\lambda_{g(i\ j)}(g, v_i)(v_{g(i)} - v_{g(j)}) = \lambda_{g(i\ j)}(g, v_j)(v_{g(j)} - v_{g(i)}),$$

apply (a) to each side and note that $(\alpha_{ij} - {}^g\alpha_{ij})(v_{g(i)} - v_{g(j)}) = (\alpha_{ji} - {}^g\alpha_{ji})(v_{g(j)} - v_{g(i)})$. $\square$

6. Deforming commutativity

Again we consider $G = \mathfrak{S}_n$ acting by permutation of basis vectors $v_1, \ldots, v_n$ of $V \cong \mathbb{F}^n$. Working toward the classification of the Drinfeld Hecke algebras $\mathcal{H}_{\lambda,\kappa}$ in Section 7, we described the form of λ in a PBW deformation in the last section and we describe the form of κ here. We again take $n > 2$ in this section (see Section 8 for $n = 1, 2, 3$) and fix linear parameter functions

$$\kappa : V \otimes V \to \mathbb{F}G \quad \text{and} \quad \lambda : \mathbb{F}G \otimes V \to \mathbb{F}G, \quad \text{with } \kappa \text{ alternating.}$$

We use the PBW Conditions (1)–(5) of Theorem 3.2. The next lemma measures the failure of κ to be invariant and follows from Theorem 5.2.

Lemma 6.1. *Assume PBW Conditions (1) and (3) hold for the parameter function λ. Then PBW Condition (2) is equivalent to*

$$\kappa({}^g v_i, {}^g v_j)g - g\kappa(v_i, v_j) = \sum_{k\neq i,j} ({}^g\alpha_{ijk} - \alpha_{ijk})\, g\left((i\ j\ k) - (i\ k\ j)\right) \quad \textit{for all } g \in G,\ i \neq j.$$

Proof. We rewrite the right-hand side of PBW Condition (2) with v_i, v_j in place of u, v using Theorem 5.2. Lemma 5.7 implies all terms vanish except those with group elements $g(i\ j\ k)$ and $g(i\ k\ j)$. The coefficient of $g(i\ j\ k)$ is

$$\sum_{k\neq i,j} \Big((\alpha_{jk} - {}^g\alpha_{jk})(\alpha_{ik} - {}^g\alpha_{ij}) - (\alpha_{ij} - {}^g\alpha_{ij})(\alpha_{jk} - {}^g\alpha_{ik}) - (\alpha_{ik} - {}^g\alpha_{ik})(\alpha_{ji} - {}^g\alpha_{jk}) \Big)$$

which simplifies to ${}^g\alpha_{ijk} - \alpha_{ijk}$ since $\alpha_{ij} = -\alpha_{ji}$, etc. Likewise, the coefficient

$$\sum_{k\neq i,j} \Big((\alpha_{ji} - {}^g\alpha_{ji})(\alpha_{ik} - {}^g\alpha_{jk}) - (\alpha_{jk} - {}^g\alpha_{jk})(\alpha_{ij} - {}^g\alpha_{ik}) - (\alpha_{ik} - {}^g\alpha_{ik})(\alpha_{jk} - {}^g\alpha_{ji}) \Big)$$

of $g(i\ k\ j)$ simplifies to $-({}^g\alpha_{ijk} - \alpha_{ijk})$ and we obtain the right-hand side of the equation in the statement. □

Lemma 6.2. *If $\mathcal{H}_{\lambda,\kappa}$ satisfies the PBW property, then for all distinct i, j, k,*

$$\kappa_{(i\ j\ k)}(v_i, v_j) = \kappa_{(i\ j\ k)}(v_j, v_k) = \kappa_{(i\ j\ k)}(v_k, v_i) = \kappa_{(i\ k\ j)}(v_i, v_k) = -\kappa_{(i\ k\ j)}(v_k, v_i)\,.$$

Proof. We use Theorem 3.2. PBW Condition (4) with $g = (i\ j\ k)$ implies that

$$\kappa_{(i\ j\ k)}(v_i, v_j)(v_i - v_k) + \kappa_{(i\ j\ k)}(v_j, v_k)(v_j - v_i) + \kappa_{(i\ j\ k)}(v_k, v_i)(v_k - v_j) = 0,$$

giving the first two equalities. Set $g = (i\ j)$ in Lemma 6.1 (whence ${}^g\alpha_{ijk} = \alpha_{ijk}$ and the right-hand side vanishes), and consider the coefficient of $(j\ k)$ to deduce the third equality. For the final equality, recall that κ is alternating. □

Proposition 6.3. *If $\mathcal{H}_{\lambda,\kappa}$ satisfies the PBW property, then κ is supported on 3-cycles, and*

$$\kappa(v_i, v_j) = \sum_{k\neq i,j} \kappa_{(i\ j\ k)}(v_i, v_j)\big((i\ j\ k) - (i\ k\ j)\big) \qquad \textit{for } i \neq j\,.$$

Proof. We first show that κ is supported on 3-cycles using Theorem 3.2 and Lemma 6.1. By Corollary 3.3, $\kappa_g \equiv 0$ unless g is the identity, a transposition, the product of two disjoint transpositions, or a 3-cycle. Assume some $\kappa_g(v_i, v_j) \neq 0$ (so $i \neq j$). Corollary 3.3 in fact implies that g must be

$$1_G,\ (i\ j),\ (j\ k),\ (i\ j)(k\ l),\ (i\ k)(j\ l),\ (i\ j\ k),\ \text{or}\ (i\ k\ j) \quad \text{for some } k \neq l \text{ and } k, l \notin \{i, j\}.$$

Let $g = (i\ j)$ in Lemma 6.1 and equate the coefficients of 1_G to conclude that

$$\kappa_{(i\ j)}(v_j, v_i) - \kappa_{(i\ j)}(v_i, v_j) = 0,$$

which implies $\kappa_{(i\ j)}(v_i, v_j) = 0$ as κ is alternating and $\text{char}(\mathbb{F}) \neq 2$. Similarly, we equate the coefficients of $(i\ j)$ to deduce that $\kappa_1(v_i, v_j) = 0$ and the coefficients of $(k\ l)$ to deduce that $\kappa_{(i\ j)(k\ l)}(v_i, v_j) = 0$. Likewise, set $g = (i\ j)(k\ l)$ and equate coefficients of $(i\ l)(j\ k)$ to see that $\kappa_{(i\ k)(j\ l)}(v_i, v_j) = 0$. To verify that $\kappa_{(j\ k)}(v_i, v_j) = 0$, on one hand we notice that $\kappa_{(j\ k)}(v_i, v_j) = \kappa_{(j\ k)}(v_i, v_k)$ after setting $g = (j\ k)$ and equating the coefficients of 1_G, and on the other hand we notice that $\kappa_{(j\ k)}(v_i, v_j) = -\kappa_{(j\ k)}(v_i, v_k)$ by PBW Condition (4) with $g = (j\ k)$ and vectors v_i, v_j, and v_k. Hence by Lemma 6.2,

$$\begin{aligned}\kappa(v_i, v_j) &= \sum_{k \neq i,j} \kappa_{(i\ j\ k)}(v_i, v_j)(i\ j\ k) + \kappa_{(i\ k\ j)}(v_i, v_j)(i\ k\ j)\\ &= \sum_{k \neq i,j} \kappa_{(i\ j\ k)}(v_i, v_j)\big((i\ j\ k) - (i\ k\ j)\big).\end{aligned}$$

□

The next proposition gives an explicit formula for $\kappa(v_i, v_j)$ when $\mathcal{H}_{\lambda,\kappa}$ is PBW.

Proposition 6.4. *If $\mathcal{H}_{\lambda,\kappa}$ satisfies the PBW property, then*

$$\kappa(v_i, v_j) = \sum_{k \neq i,j} \big(\alpha_{ijk} + \kappa_{(1\ 2\ 3)}(v_1, v_2) - \alpha_{123}\big)\big((i\ j\ k) - (i\ k\ j)\big) \quad \textit{for } i \neq j.$$

Proof. By Theorem 3.2, we may use Proposition 6.3 and Lemma 6.1 to write $g\kappa(v_i, v_j)$ in two ways and then equate the coefficients of $g(i\ j\ k)$ for distinct i, j, k in $\{1, \ldots, n\}$:

$$\kappa_{(i\ j\ k)}(v_i, v_j) = \kappa_{g(i\ j\ k)g^{-1}}(v_{g(i)}, v_{g(j)}) - {}^g\alpha_{ijk} + \alpha_{ijk}.$$

In particular, for $g = (i\ 1)(2\ j)(3\ k)$, $\kappa_{(i\ j\ k)}(v_i, v_j) = \kappa_{(1\ 2\ 3)}(v_1, v_2) - \alpha_{123} + \alpha_{ijk}$, and Proposition 6.3 implies the result. □

Corollary 6.5. *If κ is defined as in Proposition 6.4, then for all distinct i, j, k,*

$$\kappa_{(i\ j\ k)}(v_i, v_j) = \kappa_{(i\ j\ k)}(v_j, v_k) = \kappa_{(i\ j\ k)}(v_k, v_i) = \kappa_{(i\ k\ j)}(v_i, v_k) = -\kappa_{(i\ k\ j)}(v_k, v_i)\,.$$

7. Classification

We now give the classification of Drinfeld Hecke algebras for the group $G = \mathfrak{S}_n$ acting on $V \cong \mathbb{F}^n$ by permuting basis vectors $v_1, \ldots, v_n$ of V. Recall that $2 \neq \text{char}(\mathbb{F}) \geq 0$. We assume $n > 2$ in this section for ease with notation; see Section 8 for the cases $n = 1, 2, 3$. We show that every Drinfeld Hecke algebra has relations of a particular form based on parameters a_{ij}, b_k and c in $\mathbb{F}$.

Definition 7.1. *For any ordered tuple of scalars in* $\mathbb{F}$*,*

$$\mu = (a_{ij}, b_k, c:\ 1 \le i < j \le n,\ 1 \le k < n),$$

let $\mathcal{H}_\mu$ *be the* $\mathbb{F}$*-algebra generated by* V *and* $\mathbb{F}G$ *with relations*

$$gv_i - {}^g v_i g = \sum_{k=0}^{g(i)-i+n-1} b_{i+k}\, g + \sum_{j\neq i} (a_{ij} - a_{g(i)\,g(j)})\ g(i\ j) \text{ for } g \in G, 1 \le i \le n, \tag{7.2}$$

$$v_i v_j - v_j v_i = \sum_{k\neq i,j} (c - a_{123} + a_{ijk})\big((i\ j\ k) - (i\ k\ j)\big) \quad \text{for } 1 \le i \neq j \le n \tag{7.3}$$

where $a_{ijk} = a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij}$*,* $a_{ji} = -a_{ij}$*, and* $b_n = -(b_1 + \cdots + b_{n-1})$ *with indices on* b *taken modulo* n*.*

We show that the above algebras $\mathcal{H}_\mu$ make up the complete set of Drinfeld Hecke algebras:

Theorem 7.4. *For any Drinfeld Hecke algebra* $\mathcal{H}_{\lambda,\kappa}$ *for* $G = \mathfrak{S}_n$ *acting on* $V \cong \mathbb{F}^n$ *by permutations, there is an ordered tuple* $\mu = (a_{ij}, b_k, c : 1 \le i < j \le n, 1 \le k < n)$ *of scalars so that* $\mathcal{H}_{\lambda,\kappa} = \mathcal{H}_\mu$*. Conversely, for any choice* μ *of scalars,* $\mathcal{H}_\mu$ *is a Drinfeld Hecke algebra.*

Proof. First assume $\mathcal{H}_{\lambda,\kappa}$ satisfies the PBW property for some parameters λ and κ. Let $\mathcal{H}_\mu$ be the algebra of Definition 7.1 for $\mu = (a_{ij}, b_k, c)$ defined by

$$\begin{aligned}
a_{ij} &= \tfrac{1}{4}\lambda_1\big((i\ j), v_i - v_j\big) && \text{for } 1 \le i < j \le n,\\
b_k &= \tfrac{1}{2}\lambda_{(k\ k+1)}\big((k\ k+1), v_k - v_{k+1}\big) && \text{for } 1 \le k < n,\\
c &= \kappa_{(1\ 2\ 3)}(v_1, v_2), && \text{and}\\
a_{ijk} &= a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij}.
\end{aligned}$$

Using Theorem 3.2, we replace α_{ij} and β_k with a_{ij} and b_k, respectively, in Theorem 5.2 to see that λ defines the right-hand side of relation Eq. (7.2). Lemma 6.1 implies that

$$\kappa_{g(i\ j\ k)g^{-1}}(v_{g(i)}, v_{g(j)}) - \kappa_{(i\ j\ k)}(v_i, v_j) = {}^g a_{ijk} - a_{ijk} \quad \text{for all } g \in G.$$

Let $g = (1\ i)(2\ j)(3\ k)$ to see that

$$\kappa_{(1\ 2\ 3)}(v_1, v_2) - \kappa_{(i\ j\ k)}(v_i, v_j) = a_{123} - a_{ijk}.$$

Proposition 6.3 then implies that κ gives the right-hand side of Eq. (7.3). Thus $\mathcal{H}_{\lambda,\kappa} = \mathcal{H}_\mu$.

Conversely, fix an algebra $\mathcal{H}_\mu$ and set $\lambda(g, v_i)$ equal to the right-hand side of Eq. (7.2) and set $\kappa(v_i, v_j)$ equal to the right-hand side of Eq. (7.3) for all $1 \le i \neq j \le n$ and g in G. Extend λ and κ to linear parameter functions $\lambda : \mathbb{F}G \otimes V \to \mathbb{F}G$, $\kappa : V \otimes V \to \mathbb{F}G$ so that $\mathcal{H}_\mu = \mathcal{H}_{\lambda,\kappa}$.

We show that $\mathcal{H}_{\lambda,\kappa}$ is a Drinfeld Hecke algebra by verifying the five PBW Conditions of Theorem 3.2. Results from previous sections (with $\alpha_{ij} = a_{ij}$ and $\beta_k = b_k$) apply. Theorem 5.2 implies that PBW Conditions (1) and (3) hold. PBW Condition (2) is equivalent to the equation in Lemma 6.1; we examine the coefficient of each $g(i\ j\ k)$ and find that

$${}^g\kappa_{(i\ j\ k)}(v_i, v_j) - \kappa_{(i\ j\ k)}(v_i, v_j) = ({}^g a_{ijk} + c - a_{123}) - (a_{ijk} + c - a_{123}) = {}^g a_{ijk} - a_{ijk}$$

as desired. The coefficients of $g(i\ k\ j)$ are likewise equal in this equation and every other term trivially vanishes, giving PBW Condition (2).

PBW Condition (4) is trivial except for $g = (i\ j\ k)$ and v_i, v_j and v_k. By Corollary 6.5,

$$\begin{aligned}
&\kappa_{(i\ j\ k)}(v_i - v_j)({}^g v_k, v_k) + \kappa_{(i\ j\ k)}(v_j - v_k)({}^g v_i, v_i) + \kappa_{(i\ j\ k)}(v_k - v_i)({}^g v_j, v_j)\\
&\quad = \kappa_{(i\ j\ k)}(v_i, v_j)(v_i - v_k) + \kappa_{(i\ j\ k)}(v_j, v_k)(v_j - v_i) + \kappa_{(i\ j\ k)}(v_k, v_i)(v_k - v_j) = 0.
\end{aligned}$$

To verify PBW Condition (5) with v_i, v_j, and v_k, it suffices to consider the coefficients of three group elements, namely, $(i\ k\ j\ m)$, $(i\ k)$, and $(i\ j\ k)$ with indices distinct, as other terms all vanish. The coefficient of $(i\ k\ j\ m)$ vanishes: one need only expand

$$\begin{aligned}
&\kappa_{(i\ j\ m)}(v_i, v_j)(a_{ki} - a_{kj}) - \kappa_{(i\ k\ j)}(v_i, v_k)(a_{jm} - a_{im}) \\
&\quad - \kappa_{(j\ k\ m)}(v_j, v_k)(a_{im} - a_{ik}) - \kappa_{(i\ m\ k)}(v_k, v_i)(a_{jk} - a_{jm}) \\
&= (c - a_{123} + a_{ijm})(a_{ki} - a_{kj}) - (c + -a_{123} + a_{ikj})(a_{jm} - a_{im}) \\
&\quad - (c - a_{123} + a_{jkm})(a_{im} - a_{ik}) - (c - a_{123} + a_{imk})(a_{jk} - a_{jm}) \\
&= a_{ki}(a_{ijm} - a_{jkm}) + a_{kj}(a_{imk} - a_{ijm}) + a_{jm}(a_{imk} - a_{ikj}) + a_{mi}(a_{jkm} - a_{ikj}) = 0.
\end{aligned}$$

The coefficient of $(i\ k)$ is just

$$\kappa_{(i\ j\ k)}(v_i, v_j)\big((a_{ij} - a_{jk}) + (a_{ji} - a_{kj}) - (a_{kj} - a_{ji}) - (a_{jk} - a_{ij})\big) = 0.$$

The coefficient of $(i\ j\ k)$ vanishes as well by Lemma 5.6:

$$\kappa_{(i\ j\ k)}(v_i, v_j) \cdot \lambda_{(i\ j\ k)}\big((i\ j\ k), v_i + v_j + v_k\big) = 0.$$

Hence $\mathcal{H}_\mu$ satisfies the PBW property by Theorem 3.2.

□

Corollary 7.5. *For $n > 2$, the Drinfeld Hecke algebras for $\mathfrak{S}_n$ acting on $\mathbb{F}^n$ constitute a family defined by $\frac{1}{2}(n^2 + n)$ parameters in $\mathbb{F}$.*

8. Drinfeld Hecke algebras for the symmetric group in low dimensions

We now give the Drinfeld Hecke algebras more explicitly for $G = \mathfrak{S}_n$ acting in dimensions $n \leq 3$ by permuting basis vectors $v_1, \ldots, v_n$ of $V \cong \mathbb{F}^n$. They all arise from an invariant parameter κ.

One dimension. The Drinfeld Hecke algebras $\mathcal{H}_{\lambda,\kappa}$ for $n = 1$ are all trivial: Theorem 3.2 forces $\kappa \equiv 0$ and $\lambda \equiv 0$ and

$$\mathcal{H}_{\lambda,\kappa} = \mathcal{H}_{0,0} = \mathbb{F}[v_1, \ldots, v_n]\#G.$$

Two dimensions. The Drinfeld Hecke algebras $\mathcal{H}_{\lambda,\kappa}$ for $n = 2$ constitute a 2-parameter family given by

$$\begin{aligned}
\mathcal{H}_{a,b} = \mathbb{F}\langle v, g : v \in V\rangle / \big(g^2 - 1,\ \ & v_1 v_2 - v_2 v_1, \\
& (1\ 2)v_1 - v_2(1\ 2) - a - b(1\ 2), \\
& (1\ 2)v_2 - v_1(1\ 2) + a + b(1\ 2)\big),
\end{aligned}$$

for arbitrary scalars a, b in $\mathbb{F}$. This follows from Theorem 3.2 which forces $\lambda((1\ 2), v_1) = -\lambda((1\ 2), v_2)$ and $\kappa(v_1, v_2) = 0$ as $\kappa(v_1, v_2) = \kappa(v_2, v_1)$ with $\mathrm{char}(\mathbb{F}) \neq 2$.

Remark 8.1. Note that if we were to allow $\mathrm{char}(\mathbb{F}) = 2 = n$, we would find instead a 4-parameter family of Drinfeld Hecke algebras: for arbitrary a, b, c, d in $\mathbb{F}$,

$$\begin{aligned}
\mathcal{H}_{a,b,c,d} = \mathbb{F}\langle v, g : v \in V\rangle / \big(g^2 - 1,\ \ & v_1 v_2 - v_2 v_1 - c - d(1\ 2), \\
& (1\ 2)v_1 - v_2(1\ 2) - a - b(1\ 2), \\
& (1\ 2)v_2 - v_1(1\ 2) - a - b(1\ 2)\big).
\end{aligned}$$

Three dimensions. The Drinfeld Hecke algebras $\mathcal{H}_{\lambda,\kappa}$ for $n = 3$ constitute a 6-parameter family:

Proposition 8.2. *Let $\lambda : \mathbb{F}G \otimes V \to \mathbb{F}G$ and $\kappa : V \otimes V \to \mathbb{F}G$ be linear parameter functions with κ alternating. The algebra $\mathcal{H}_{\lambda,\kappa}$ generated by a basis v_1, v_2, v_3 of V and $\mathbb{F}G$ with relations*

$$v_i v_j - v_j v_i = \kappa(v_i, v_j) \quad \text{and} \quad g v_i - {}^g v_i g = \lambda(g, v_i) \quad \text{for } g \in G$$

satisfies the PBW property if and only if there are scalars $a_1, a_2, a_3, b_1, b_2, c$ in $\mathbb{F}$ with

$$\kappa(v_i, v_j) = c\big((i\ j\ k) - (i\ k\ j)\big) \quad \text{for all } i, j, k \ \text{ distinct}$$

and λ is defined by

$$\begin{aligned}
&\lambda((1\ 2), v_1) = a_1 + b_1(1\ 2) - (a_2 + a_3)(1\ 3\ 2), && \lambda((2\ 3), v_1) = (a_1 + a_3)\big((1\ 3\ 2) + (1\ 2\ 3)\big),\\
&\lambda((1\ 2), v_2) = -a_1 - b_1(1\ 2) + (a_2 + a_3)(1\ 2\ 3), && \lambda((2\ 3), v_2) = a_2 + b_2(2\ 3) - (a_1 + a_3)(1\ 3\ 2),\\
&\lambda((1\ 2), v_3) = (a_2 + a_3)\big((1\ 3\ 2) - (1\ 2\ 3)\big), && \lambda((2\ 3), v_3) = 1 + (2\ 3) - (1\ 2\ 3),\\
&\lambda((1\ 3), v_2) = (a_1 + a_2)\big((1\ 3\ 2) - (1\ 2\ 3)\big), && \lambda((1\ 3), v_1) = -a_3 - b_3(1\ 3) + (a_1 + a_2)(1\ 2\ 3),\\
&\lambda((1\ 3), v_3) = a_3 - (a_1 + a_2)(1\ 3\ 2) + b_3(1\ 3), &&
\end{aligned}$$

$$\begin{aligned}
\lambda((1\ 2\ 3), v_1) &= (a_1 - a_2)(1\ 3) - (a_3 + a_2)(2\ 3) + b_1(1\ 2\ 3),\\
\lambda((1\ 2\ 3), v_2) &= (a_2 + a_3)(1\ 2) + (a_2 - a_1)(1\ 3) + b_2(1\ 2\ 3),\\
\lambda((1\ 2\ 3), v_3) &= (a_2 + a_3)(2\ 3) - (a_2 + a_3)(1\ 2) + b_3(1\ 2\ 3),\\
\lambda((1\ 3\ 2), v_1) &= (a_2 - a_3)(1\ 2) + (a_1 - a_3)(2\ 3) - b_3(1\ 3\ 2),\\
\lambda((1\ 3\ 2), v_2) &= (a_3 - a_1)(2\ 3) + (a_2 - a_1)(1\ 3) - b_2(1\ 3\ 2),\\
\lambda((1\ 3\ 2), v_3) &= (a_1 - a_2)(1\ 3) + (a_3 - a_2)(1\ 2) - b_1(1\ 3\ 2)\,,
\end{aligned}$$

where $b_3 = -(b_1 + b_2)$.

Proof. Theorem 7.4 implies that $\kappa_{(i\ j\ k)}(v_i, v_j)$ is determined by $a_{ijk} - a_{123}$, but for distinct i, j, k, $a_{ijk} = a_{123}$ since $a_{ij} = -a_{ji}$, which implies that κ is defined as indicated. The parameter λ is defined as in Theorem 7.4 with $a_1 = a_{12}, a_2 = a_{23}$, and $a_3 = a_{31}$. □

9. Commutativity up to an invariant parameter

We saw in Section 4 that Drinfeld Hecke algebras in the nonmodular setting all arise from a parameter κ which is *G-invariant*. Again, let $G = \mathfrak{S}_n$ act on $V \cong \mathbb{F}^n$ by permuting basis vectors $v_1, \ldots, v_n$ of V. In Section 8, we observed that the Drinfeld Hecke algebras in the modular setting in low dimension ($n = 1, 2, 3$) also all arise from a parameter κ which is G-invariant. By Theorem 7.4, other Drinfeld Hecke algebras $\mathcal{H}_{\lambda,\kappa}$ with more general parameters λ and κ arise, but here we investigate those with κ invariant. We assume $n > 2$ in this section. Note that by PBW Condition (2), the invariance of κ forces

$$0 = \lambda\big(\lambda(g, v), u\big) - \lambda\big(\lambda(g, u), v\big) \quad \text{for} \quad u, v \in V,\ g \in G\,.$$

Again, we take indices on b modulo n.

Corollary 9.1. *An algebra $\mathcal{H}_{\lambda,\kappa}$ satisfies the PBW property with κ invariant if there are scalars c, d, a_{1i}, b_j in $\mathbb{F}$ for $1 < i \leq n$, $1 \leq j < n$, with the a_{1i} distinct, such that*

$$\kappa(v_i, v_j) = c \sum_{k\neq i,j} (i\ j\ k) - (i\ k\ j) \qquad \text{for} \quad 1 \leq i \neq j \leq n, \quad \text{and}$$

$$\lambda(g, v_i) = \sum_{k=0}^{g(i)-i+n-1} b_{i+k}\, g \ + \sum_{j\neq i} (a_{ij} - a_{g(i)g(j)})\, g(i\ j) \qquad \text{for} \quad 1 \leq i \leq n\,,$$

where $a_{ij} = \frac{d + a_{1i}a_{1j}}{a_{1i} - a_{1j}}$ for $i, j \neq 1$, $i \neq j$, $a_{1i} = -a_{i1}$, and $b_n = -(b_1 + \cdots + b_{n-1})$.

Proof. Consider the ordered tuple $\mu = (a_{ij}, b_k, c : 1 \leq i < j \leq n, 1 \leq k < n)$ and let $\mathcal{H}_\mu$ be the PBW algebra of Theorem 7.4. A calculation confirms that $a_{ijk} = a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij} = d$ for all distinct i, j, k and thus $a_{ijk} - a_{123} = 0$, implying that $\mathcal{H}_\mu = \mathcal{H}_{\lambda,\kappa}$. Thus $\mathcal{H}_{\lambda,\kappa}$ satisfies the PBW property and one may check the invariance of κ directly. $\square$

Proposition 9.2. *The algebra $\mathcal{H}_{\lambda,0}$ satisfies the PBW property for $\lambda : \mathbb{F}G \otimes V \to \mathbb{F}G$ defined up to scalar in $\mathbb{F}$ by*

$$\lambda(g, v_i) = (g(i) - i)\, g \qquad \text{for all } g \in G,\ 1 \leq i \leq n.$$

Proof. In Theorem 7.4, set $a_{ij} = 0$, $c = 0$, and $b_k = 1$ for all $1 \leq i < j \leq n$, $1 \leq k < n$. Then κ vanishes and $\lambda(g, v)$ is supported on g with

$$\lambda(g, v_i) = \sum_{k=0}^{g(i)-i+n-1} b_i\, g = (g(i) - i)\, g,$$

as $b_n = -(n-1)$. For a scalar multiple, just choose the same constant for all b_k, $k < n$. $\square$

Note that the converse of Proposition 9.2 fails: There are examples of PBW algebras $\mathcal{H}_{\lambda,0}$ with λ taking other forms.

Corollary 9.3. *If $\mathcal{H}_{\lambda,0}$ satisfies the PBW property, then so does $\mathcal{H}_{\lambda,\kappa}$ for*

$$\kappa : V \otimes V \to \mathbb{F}G \qquad \text{defined by} \quad \kappa(v_i, v_j) = \sum_{k\neq i,j} (i\ j\ k) - (i\ k\ j) \qquad \text{for} \quad i \neq j.$$

Proof. By Theorem 7.4, there is an ordered tuple $\mu = (a_{ij}, b_k, c : 1 \leq i < j \leq n, i \leq k < n)$ so that $\mathcal{H}_\mu = \mathcal{H}_{\lambda,0}$. Observe that $c = a_{123} - a_{ijk}$ for all distinct i, j, k. Set $c' = c + 1$ and consider the ordered tuple $\mu' = (a_{ij}, b_k, c' : 1 \leq i < j \leq n, i \leq k < n)$. The algebra $\mathcal{H}_{\mu'}$ satisfies the PBW property by Theorem 7.4. Since $c' - a_{123} + a_{ijk} = 1$ for all distinct i, j, k, $\mathcal{H}_{\mu'} = \mathcal{H}_{\lambda,\kappa}$ for κ as given in the statement, and $\mathcal{H}_{\lambda,\kappa}$ satisfies the PBW property. $\square$

We end by highlighting a handy 2-parameter family of Drinfeld Hecke algebras.

Corollary 9.4. *The algebra generated by v in $V = \mathbb{F}^n$ and $\mathbb{F}G$ with relations*

$$gv_i - {}^g v_i\, g = a\big(g(i) - i\big)g \quad \text{and} \quad v_iv_j - v_jv_i = b \sum_{k\neq i,j} (i\ j\ k) - (i\ k\ j) \qquad \text{for all } g \in G, i \neq j$$

satisfies the PBW property for any a, b in $\mathbb{F}$.

Proof. Use Theorem 7.4 with $a_{ij} = 0$, $c = 1$, and $b_k = 1$ for all $1 \leq i < j \leq n$, $1 \leq k < n$. $\square$

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