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JACOBIANS OF REFLECTION GROUPS

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ABSTRACT. Steinberg showed that when a finite reflection group acts on a real or complex vector space of finite dimension, the Jacobian determinant of a set of basic invariants factors into linear forms which define the reflecting hyperplanes. This result generalizes verbatim to fields whose characteristic is prime to the order of the group. Our main theorem gives a generalization of Steinberg's result for groups with polynomial ring of invariants over arbitrary fields using a ramification formula of Benson and Crawley-Boevey.

1. INTRODUCTION

The advent of modern algebra owes much to invariant theory. Many of our classical theorems arose from Noether's investigations of a finite group $G \leq \mathrm{Gl}(V)$ acting linearly on an n-dimensional vector space V over a field $\mathbb{F}$. The action of G induces a natural action on the polynomial ring $\mathbb{F}[V] \cong \mathrm{Sym}(V^*)$. Noether showed that the ring $\mathbb{F}[V]^G$ of invariant polynomials is finitely generated as an algebra. We are interested in the case when generators of $\mathbb{F}[V]^G$ are algebraically independent: we say that G has a **polynomial ring of invariants** if $\mathbb{F}[V]^G = \mathbb{F}[f_1, \ldots, f_n]$ for some homogeneous polynomials f_i called **basic invariants**. Although there are many choices of basic invariants, their degrees are unique, and thus the integers $\deg f_1 - 1, \ldots, \deg f_n - 1$ depend only on the group. We call these integers the **exponents** of G. When G has a polynomial ring of invariants, we define the **Jacobian determinant** $J = J(f_1, \ldots, f_n) = \det(\partial f_i / \partial z_j)$. This polynomial is nonzero (see [Ben93, 5.4]) and well-defined up to a nonzero element of $\mathbb{F}$ depending on the choice of basic invariants and basis $\{z_j\}$ of V^*. In this article, we examine the structure of the Jacobian determinant J. No assumption is made on the ground field $\mathbb{F}$.

Elements of finite order in $\mathrm{Gl}(V)$ which fix a hyperplane pointwise are called **reflections**. (We consider the identity a reflection.) For any subgroup G of $\mathrm{Gl}(V)$ and hyperplane H in V, define

$$G_H = \{\sigma \in G : \sigma\big|_H = \mathrm{id}_H\},$$

the pointwise stabilizer of H in G. The hyperplanes H for which G_H is nontrivial are called **reflecting hyperplanes** of G. For each hyperplane H in V, let l_H in V^* be a linear form with $\ker l_H = H$.

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In Section 2, we examine the set of root vectors of reflections about a common hyperplane. Our analysis shows that every finite subgroup of $\mathrm{Gl}(V)$ which fixes a hyperplane H in V pointwise has a polynomial ring of invariants and the Jacobian determinant is a power of l_H.

Our main result (Theorem 3.4) states that if $G \leq \mathrm{Gl}(V)$ is a finite group with a polynomial ring of invariants, its Jacobian determinant factors as a product of linear forms defining the reflecting hyperplanes of G. The multiplicity with which each linear form l_H occurs is the sum of the exponents of G_H.

This theorem has roots in the rich theory of reflection groups. A finite subgroup of $\mathrm{Gl}(V)$ is a **reflection group** if it is generated by reflections, and the collection $\mathcal{A}$ of its reflecting hyperplanes is called the **reflection arrangement**. In the **nonmodular case**, i.e., when the characteristic of $\mathbb{F}$ is prime to $|G|$, a well-known theorem of Serre and Shephard, Todd, and Chevalley (see [Smi95, Ch. 7]) states that a finite subgroup of $\mathrm{Gl}(V)$ is a reflection group if and only if it has a polynomial ring of invariants. Steinberg [Ste60] showed in this case that the Jacobian determinant of a set of basic invariants factors into powers of linear forms defining the reflecting hyperplanes:

$$J \doteq \prod_{H \in \mathcal{A}} l_H^{|G_H|-1}$$

(we write $a \doteq b$ to indicate that a and b are equal up to a nonzero constant). In particular, the above factorization holds for all Weyl groups, Coxeter groups, and complex reflection groups (see [OT92, Thm. 6.42]). In the nonmodular case, each G_H is a cyclic group and the only nonzero exponent of G_H is $|G_H| - 1$.

Serre [Ser67] showed that in arbitrary characteristic, every finite subgroup of $\mathrm{Gl}(V)$ with a polynomial ring of invariants must be generated by reflections. The converse may fail when the characteristic of $\mathbb{F}$ divides the order of G (for example, see [KM97]). Unfortunately, Steinberg's characterization of the Jacobian determinant in terms of the integers $|G_H|$ no longer holds over arbitrary fields. The stabilizer subgroups G_H may not be cyclic, in which case the integers $|G_H| - 1$ as H runs over all reflecting hyperplanes will usually not sum to the degree of J. This is no surprise, as the class of reflections is larger in some sense over an arbitrary field than over a characteristic zero field. The reflections in $\mathrm{Gl}(V)$ not only include diagonalizable reflections (with a single nonidentity eigenvalue), but also **transvections**, reflections with determinant 1 which can not be diagonalized. The transvections in $\mathrm{Gl}(V)$ prevent one from developing a theory of reflection groups mirroring that for Coxeter groups or complex reflection groups. (For example, even if a reflection group has a polynomial ring of invariants, the Jacobian J may be invariant or lie in the Hilbert ideal generated by the basic invariants—see Section 4.) If G lacks transvections, then it shares some characteristics with reflection groups over characteristic zero fields, for example, the pointwise stabilizer of any hyperplane in V is cyclic. One can deduce that Steinberg's description remains valid in this special case (see [Har01]).

Theorem 3.4 implies a statement conjectured by Victor Reiner: if G has a polynomial ring of invariants, then the zero locus of the Jacobian determinant is exactly the union of the reflecting hyperplanes. Reiner, Stanton, and Webb [RSW04] use this corollary in generalizing Springer's theory of regular numbers in characteristic zero to arbitrary fields.

2. ONE HYPERPLANE

In this section, we consider finite groups $G \leq \mathrm{Gl}(V)$ that fix a hyperplane H in V pointwise and we investigate how the geometry of root vectors determines the group structure. It is easy to see that such groups have polynomial invariants in characterisitic zero. Landweber and Stong [LS87] prove that the same holds in nonzero characteristic. We give a constructive proof of this fact which also shows that the Jacobian determinant of a set of basic invariants defines the hyperplane.

An explicit description of the basic invariants over fields of prime order can be found in Smith [Smi95, Chap. 8], for example. Unfortunately, parts of the description (end of Section 2 in [Smi95]) do not extend to other (finite) fields (see Example 4.1 in the last section).

Let H be a hyperplane in V defined by some linear form $l_H \in V^*$. For any reflection $\tau \in \mathrm{Gl}(V)$ which fixes H pointwise, let v_τ be the **root vector** of τ (with respect to l_H) defined by

$$\tau(v) = v + l_H(v)\, v_\tau \qquad \text{for all } v \in V.$$

Note that a transvection is a reflection whose root vector lies in its reflecting hyperplane, i.e., $l_H(v_\tau) = 0$ (see for example [NS02], section 6.2). For any set S of reflections, let $\mathcal{R}(S)$ be the corresponding set of root vectors in V. (Of course $\mathcal{R}(S)$ depends on our choice of l_H.)

If $\mathrm{char}(\mathbb{F}) = 0$, then any group G which fixes a hyperplane pointwise is necessarily cyclic and its order equals the maximal order of a diagonalizable reflection in G (which then generates G). When $\mathrm{char}(\mathbb{F}) > 0$, a group G which fixes a hyperplane pointwise is a semidirect product of the normal subgroup K generated by the transvections (the kernel of the determinant map) and a cyclic subgroup generated by a diagonalizable reflection σ of maximal order. The next lemma gives the order of G in this case.

Lemma 2.1. *Assume that* $\mathrm{char}(\mathbb{F}) = p > 0$. *Suppose* $G \leq \mathrm{Gl}(V)$ *is a finite group which fixes a hyperplane* H *in* V *pointwise. Let* σ *be a diagonalizable reflection in* G *of maximal order with* $c = \det(\sigma)$. *Let* $K \trianglelefteq G$ *be the subgroup generated by the transvections in* G. *Then*

(1) *The action of* σ *on* K *by conjugation translates into multiplication by* c *on* $\mathcal{R}(K)$ *and thereby endows* $\mathcal{R}(K)$ *with the structure of an* $\mathbb{F}_p(c)$*-vector space.*
(2) $T \subset K$ *is a minimal set satisfying* $G = \langle T, \sigma\rangle$ *if and only if* $\mathcal{R}(T) \subset \mathcal{R}(K)$ *is a basis for* $\mathcal{R}(K)$ *over* $\mathbb{F}_p(c)$.
(3) *The group* G *has order* $|c| \cdot |\mathbb{F}_p(c)|^d$, *where* d *is the minimal number of transvections needed to generate* G *together with* σ.

Proof. If ρ and τ are both reflections about H, then the root vector of the product is a linear combination of the root vectors:

$$(*) \qquad v_{\rho\tau} = c_\tau\, v_\rho + v_\tau,$$

where $c_\tau = 1 + l_H(v_\tau)$ is the nonidentity eigenvalue of τ (if τ is not a transvection) or the eigenvalue 1 (if τ is a transvection). In particular, $v_{\rho\tau} = v_\rho + v_\tau$ for all $\rho \in G$, $\tau \in K$. Note that $v_\sigma = -c v_{\sigma^{-1}}$. Fix some transvection τ in K. An easy computation then shows that

$$(**) \qquad v_{\sigma^{-1}\tau\sigma} = c v_\tau \quad \text{and thus} \quad v_{\sigma^{-1}\tau^m\sigma} = c v_{\tau^m} = (mc) v_\tau$$

for $m \in \{0, 1, \ldots, p-1\}$.

We claim that the root vector of any element in the subgroup $\langle \sigma, \tau \rangle$ must lie in the $\mathbb{F}_p(c)$-span of v_σ and v_τ. Indeed, we can write the element as a product of the generators σ and τ and use Equation $(*)$ repeatedly. In particular, the root vector of a transvection in $\langle \sigma, \tau \rangle$ lies on H and thus must be an $\mathbb{F}_p(c)$-multiple of v_τ alone (as v_τ lies on H but v_σ does not). On the other hand, any $\mathbb{F}_p(c)$-multiple of v_τ is the root vector of some transvection in $\langle \sigma, \tau \rangle$: if $e = |c|$ and

$$v = (m_1 c + m_2 c^2 + \ldots + m_e c^e)\ v_\tau \quad \text{for some } m_i \in \{0, 1, \ldots, p-1\},$$

then v is the root vector of the transvection

$$(\sigma^1 \tau^{m_1} \sigma^{-1})(\sigma^2 \tau^{m_2} \sigma^{-2}) \cdots (\sigma^e \tau^{m_e} \sigma^{-e})$$

in $\langle \sigma, \tau \rangle$ by Equation $(**)$. Since each transvection about H is determined by its root vector, the transvections in $\langle \sigma, \tau \rangle$ correspond bijectively to the $\mathbb{F}_p(c)$-multiples of v_τ.

More generally, if $\tau_1, \ldots, \tau_k$ are transvections in G, then a similar argument shows that the $\mathbb{F}_p(c)$-span of $v_{\tau_1}, \ldots, v_{\tau_k}$ is the set of root vectors corresponding to the group $K \cap \langle \sigma, \tau_1, \ldots \tau_k \rangle$. This proves part (1).

Suppose $T = \{\tau_1, \ldots, \tau_k\}$ is a minimal subset of K satisfying $G = \langle T, \sigma \rangle$ and let $v_i = v_{\tau_i}$. Then no τ_j lies in the group generated by σ and $\{\tau_i : i \neq j\}$ and hence no v_j is an $\mathbb{F}_p(c)$-linear combination of $\{v_i : i \neq j\}$. Thus, the root vectors $v_1, \ldots, v_k$ are linearly independent over $\mathbb{F}_p(c)$. As T generates G together with σ, the root vectors $v_1, \ldots, v_k$ span $\mathcal{R}(K)$ over $\mathbb{F}_p(c)$ and hence form a basis. Conversely, if the root vectors of some set T form an $\mathbb{F}_p(c)$-basis of $\mathcal{R}(K)$, then T is a minimal subset of K generating G together with σ, which proves (2).

Finally, if $\mathcal{R}(T)$ is a basis of $\mathcal{R}(K)$ over $\mathbb{F}_p(c)$ for some $T \subset K$, then

$$|K| = |\mathcal{R}(K)| = |\operatorname{span}_{\mathbb{F}_p(c)} \mathcal{R}(T)| = |\mathbb{F}_p(c)|^{|T|},$$

and hence $|G| = |\sigma| \cdot |K| = |c| \cdot |\mathbb{F}_p(c)|^{|T|}$. □

Recall that a polynomial $f \in \mathbb{K}[x_1, \ldots, x_r]$ over a field $\mathbb{K}$ is called **additive** if it induces an additive homomorphism $\mathbb{K}^r \to \mathbb{K}$. The following lemma is needed in the proof of Proposition 2.3 to inductively construct basic invariants.

Lemma 2.2. *Let $\mathbb{F}$ be a field and let $A \subseteq \mathbb{F}$ be a finite additive subgroup. Then the polynomial $f(X) = \prod_{a \in A} (X + a) \in \mathbb{F}[X]$ is additive.*

Proof. Consider the polynomial

$$F(X, t) := f(X + t) - f(X) = a_{m-1}(t) X^{m-1} + \ldots + a_1(t) X + a_0(t),$$

where t is another variable and the a_i are polynomials in t. Note that $a_0(t) = F(0, t) = f(t)$ by definition, and $\deg_t(a_i) < m$ for $i \geq 1$. But $F(X, t_0) = 0$ for all $t_0 \in A$, since $f(X + t_0) = f(X)$ by definition of f. Hence, for every $t_0 \in A$, the coefficients $a_1(t_0), \ldots, a_{m-1}(t_0)$ are all zero. Thus for $i \geq 1$, the polynomial $a_i(t)$ has at least m zeroes. Since each a_i has degree at most $m - 1$ in t, it must be identically zero. This shows that

$$f(X + t) - f(X) = F(X, t) = a_0(t) = f(t),$$

and so f is additive. □

The reader who is familiar with the Landweber-Stong invariants over prime fields is encouraged to peruse Example 4.1 in the last section before considering the next proposition and its proof.

Proposition 2.3. *Let $H \leq V$ be a hyperplane defined by some $l_H \in V^*$. Let $G \leq \mathrm{Gl}(V)$ be a finite group fixing H pointwise. Then*
1. *The group G has a polynomial ring of invariants.*
2. *The Jacobian determinant is $J \doteq l_H^m$, where m is the sum of the exponents of G.*

Proof. The group G is generated by a diagonalizable reflection σ with eigenvalue c of order e together with a minimal set of transvections $\tau_1, \ldots, \tau_r$ (see Lemma 2.1). For $k = 1, \ldots, r$, let $G_k = \langle \sigma, \tau_1, \ldots, \tau_k \rangle$, and let $G_0 = \langle \sigma \rangle$. Choose a basis $e_1, \ldots, e_n$ of V such that σ is in diagonal form and $e_1, \ldots, e_{n-1}$ span H. Let $z_1, \ldots, z_n$ be the dual basis of V^* and rescale l_H so that $z_n = l_H$. Consider the case $\mathrm{char}(\mathbb{F}) = p > 0$.

We prove by induction on k a stronger statement: $\mathbb{F}[V]^G = \mathbb{F}[f_1, \ldots, f_n]$ for some homogeneous polynomials f_i where $f_n = z_n^e$, $J(f_1, \ldots, f_n) = z_n^m$, and for $i < n$, the degree of f_i is a p-power and f_i is additive as a polynomial in $\mathbb{F}(z_n)[z_1, \ldots, z_{n-1}]$. For $k = 0$, these claims are satisfied by setting $f_n = z_n^e$ and $f_i = z_i$ for $i < n$. Note that these are also the basic invariants when the characteristic of $\mathbb{F}$ is zero (as $G = \langle \sigma \rangle$ in this case).

Let $k \geq 0$ and assume the induction hypothesis holds for the group G_k with $\mathbb{F}[V]^{G_k} = \mathbb{F}[f_1, \ldots, f_n]$. Let d_i be the degree of each f_i and let $\tau = \tau_{k+1}$. By our choice of basis, $\tau(z_i) = z_i + a_i z_n$ for some $a_i \in \mathbb{F}$ when $i < n$ and $\tau(z_n) = z_n$. For $i < n$, each f_i is additive over the infinite field $\mathbb{F}(z_n)$, and thus

$$\begin{aligned} \tau f_i(z_1, \ldots, z_n) &= f_i(z_1 + a_1 z_n, \ldots, z_{n-1} + a_{n-1} z_n, z_n) \\ &= f_i(z_1, \ldots, z_n) + f_i(a_1 z_n, \ldots, a_{n-1} z_n, z_n) = f_i + b_i z_n^{d_i} \end{aligned}$$

for some $b_i \in \mathbb{F}$ (note that the second summand only depends on the variable z_n). Thus $b_i = 0$ exactly when f_i is invariant under τ.

Relabel $f_1, \ldots, f_{n-1}$ so that f_1 has minimal degree among those f_i which are not invariant under τ. Define $f'_2, \ldots, f'_{n-1}$ by

$$f'_i = f_i + b'_i f_1^{d_i/d_1} \quad \text{where } b'_i = -b_i / b_1^{d_i/d_1}.$$

The constants b'_i are chosen so that each f'_i is invariant under τ. The degrees of $f'_2, \ldots, f'_{n-1}$ are again p-powers, since d_i/d_1 is a nonnegative p-power whenever $b'_i \neq 0$. Furthermore, $f'_2, \ldots, f'_{n-1}$ are additive over $\mathbb{F}(z_n)$ as they are the compositions of additive homomorphisms. Define $f'_n = f_n$. Then $f'_2, \ldots, f'_n$ are invariant under τ and $\sigma, \tau_1, \ldots, \tau_k$ (as each $f_i \in \mathbb{F}[V]^{G_k}$). Hence, $f'_2, \ldots, f'_n$ are invariant under G_{k+1}.

We take the product over the orbit of f_1 to produce a polynomial f'_1 invariant under τ. Define

$$h(X) = \prod_{a \in A} (X + a z_n^{d_1}) \quad \in \mathbb{F}(z_n)[X],$$

where $A = \mathbb{F}_p(c) b_1 = \{ b \cdot b_1 \mid b \in \mathbb{F}_p(c) \}$. By Lemma 2.2, $h(X)$ is additive as a polynomial in $\mathbb{F}(z_n)[X]$. Define $f'_1 = h(f_1) \in \mathbb{F}[z_1, \ldots, z_n]$. Then f'_1 is additive in $\mathbb{F}(z_n)[z_1, \ldots, z_{n-1}]$ as it is the composition of additive homomorphisms. The polynomial f'_1 is invariant under τ (by its very definition) and invariant under $\tau_1, \ldots, \tau_k$ since both f_1 and z_n are. The polynomial f'_1 is also invariant under the diagonalizable reflection σ since $\sigma(f_1) = f_1$, $\sigma(z_n) = c^{-1} z_n$, and A is closed under multiplication by $\mathbb{F}_p(c)$. (In particular, f_1' is a polynomial in f_1 and f_n.) Hence, $f_1', \ldots, f_n' \in \mathbb{F}[V]^{G_{k+1}}$.

We consider each f_i' as a polynomial in $f_1, \ldots, f_n$. By the chain rule,

$$J(f_1', \ldots, f_n') \;=\; J(f_1, \ldots, f_n)\ \det\left(\frac{\partial f_i'}{\partial f_j}\right).$$

The matrix $(\partial f_i'/\partial f_j)$ is upper triangular with determinant $\frac{\partial f_1'}{\partial f_1}$. Since h is additive as a polynomial in $\mathbb{F}(z_n)[X]$ by Lemma 2.2, every exponent of X in an expansion of h is a p-power (see [Lan02], VI §12, for example). If we expand $f_1' = h(f_1)$ as polynomial in f_1 and z_n, every exponent of f_1 will thus also be a p-power. In particular[1],

$$\partial f_1'/\partial f_1 \;=\; -b_1^{|A|-1}\ z_n^{d_1(|A|-1)},$$

and hence

$$J(f_1', \ldots, f_n') \doteq J(f_1, \ldots, f_n) \cdot z_n^{d_1(|A|-1)}.$$

By the induction hypothesis, $J(f_1, \ldots, f_n)$ is a power of z_n and the exponent of z_n is the sum of the exponents of G_k. Substituting this into the last equality shows that assertion (2) holds for G_{k+1} and thus for G by induction. The polynomials $f_1', \ldots, f_n'$ form a set of basic invariants for G_{k+1} if and only if $J(f_1', \ldots, f_n')$ is nonzero and the product of the degrees of the f_i' is the order of the group G_{k+1} (for example, see [Kem96, Prop. 16]). By Lemma 2.1 and the induction hypothesis,

$$\deg f_1' \cdots \deg f_n' = |\mathbb{F}_p(c)| \deg f_1 \deg f_2 \cdots \deg f_n = |\mathbb{F}_p(c)||G_k| = |G_{k+1}|,$$

and (1) follows. □

The proof of Proposition 2.3 shows an interesting fact. The polynomials f_1, $f_2', \ldots, f_n'$ in the induction step of the proof form a set of basic invariants for G_k. Thus, if we choose basic invariants of G_k wisely, we need only adjust *one* of them to produce basic invariants for G_{k+1}:

Corollary 2.4. *Let $G \leq \mathrm{Gl}(V)$ be a finite group which fixes a hyperplane H in V pointwise. Let $G' = \langle G, \tau \rangle$ where $\tau \notin G$ is a transvection about H. Then there exist basic invariants $f_1, \ldots, f_n$ for G and an invariant f_1' for G' such that $f_1', f_2, \ldots, f_n$ form a set of basic invariants for G' (in particular, all basic invariants for G except one can be chosen to be invariant under G').*

3. The Jacobian Factors

In this section, we consider a finite group $G \leq \mathrm{Gl}(V)$ with a polynomial ring of invariants. We show that the Jacobian determinant factors into powers of linear forms defining the reflecting hyperplanes. We begin with an easy consequence of Proposition 2.3:

Lemma 3.1. *Assume that $G \leq \mathrm{Gl}(V)$ is a finite group with a polynomial ring of invariants. Let $\mathcal{A}$ be the reflection arrangement of G. Then the Jacobian determinant is divisible by*

$$\prod_{H \in \mathcal{A}} l_H^{m_H},$$

where each m_H is the sum of the exponents of the pointwise stabilizer G_H.

[1] One can use the fact that $\mathbb{F}_p(c)$ is the splitting field of the polynomial $X^{|A|} - X$ to compute the coefficient; we do not use this coefficient in what follows.

Proof. Fix some reflecting hyperplane $H \in \mathcal{A}$. By Proposition 2.3, G_H has a polynomial ring of invariants and

$$J(f_1^H, \ldots, f_n^H) \doteq z_n{}^{m_H},$$

where $f_1^H, \ldots, f_n^H$ are basic invariants for G_H. Let $f_1, \ldots, f_n$ denote basic invariants for G. Each f_i is invariant under G_H and hence may be written as a polynomial in the f_j^H. Thus $J(f_1^H, \ldots, f_n^H) = l_H^{m_H}$ divides $J(f_1, \ldots, f_n)$ by the chain rule. The claim then follows since the linear forms l_H for different reflecting hyperplanes are pairwise coprime and $\mathbb{F}[V]$ is a unique factorization domain. □

We next verify that we have found all factors of J. We compare degrees using the following version of the ramification formula of Benson and Crawley-Boevey (Corollary 3.12.2 in [Ben93]):

Lemma 3.2. *Assume that $G \leq \mathrm{Gl}(V)$ is a finite group. Then*

$$|G|\ \psi(\mathbb{F}[V]^G) \ = \ \sum_{H \leq V} |G_H|\ \psi(\mathbb{F}[V]^{G_H})$$

(the sum runs over all hyperplanes in V). Here $\psi(M)$ denotes the coefficient of $\frac{1}{(1-t)^{n-1}}$ in the expansion at $t = 1$ of the Poincaré series of a finitely generated $\mathbb{F}[V]^G$-module M.

We apply the above lemma and obtain

Lemma 3.3. *Assume that $G \leq \mathrm{Gl}(V)$ is a finite group with a polynomial ring of invariants and let $\mathcal{A}$ be its reflection arrangement. Let J be the Jacobian determinant of G. Then*

$$\deg(J) = \sum_{H \in \mathcal{A}} m_H,$$

where each m_H is the sum of the exponents of the pointwise stabilizer G_H.

Proof. By Proposition 2.3, each G_H has a polynomial ring of invariants. The product of the degrees d_i^H of basic invariants for G_H equals the order of G_H by [Kem96, Prop. 16]. Consequently,

$$\begin{aligned}
\frac{1}{2}\deg(J) \ &= \ \frac{1}{2}\sum_{i=1}^{n}(d_i - 1) = \ |G|\ \psi(\mathbb{F}[V]^G) \\
&= \sum_H |G_H|\ \psi(\mathbb{F}[V]^{G_H}) && \text{by Lemma 3.2} \\
&= \sum_H |G_H|\ \frac{1}{2|G_H|}\ \sum_{i=1}^{n}(d_i^{(H)} - 1) = \ \frac{1}{2}\sum_H m_H.
\end{aligned}$$

which proves the claim. □

The main theorem is a direct consequence.

Theorem 3.4. *Assume that $G \leq \mathrm{Gl}(V)$ is a finite group with a polynomial ring of invariants. Then the Jacobian determinant J factors into a product of powers of linear forms defining the reflecting hyperplanes. In fact,*

$$J \ \doteq \ \prod_{H \in \mathcal{A}} l_H^{m_H},$$

where m_H denotes the sum of the exponents of the pointwise stabilizer G_H.

Proof. By Lemma 3.1, the right hand side of the equation divides J. By Lemma 3.3, both sides have the same degree. Thus they are equal up to a scalar. □

We immediately obtain

Corollary 3.5. *Assume that $G \leq \mathrm{Gl}(V)$ is a finite group with a polynomial ring of invariants. Then the zero set of the Jacobian determinant is the union of all reflecting hyperplanes of G.*

Remark 3.6. There is a geometric proof of Corollary 3.5 suggested by W. Messing: Since the extension of quotient fields $\mathrm{Quot}(\mathbb{F}[V])/\mathrm{Quot}(\mathbb{F}[V]^G)$ is separable of degree $|G|$, the associated quotient morphism $\pi : V \to V/G$ is an étale covering in a neighborhood of a point $v \in V$ if and only if G acts freely on v. By (a variant of) a theorem of Serre, this happens if and only if v avoids all reflecting hyperplanes. Since $\mathbb{F}[V]^G$ is a polynomial ring, π is a morphism of affine varieties, and thus it is étale near v if and only if the Jacobian matrix evaluated at v is invertible, i.e., has nonzero determinant. See [RSW04] for the full argument.

4. Examples

We first give an example to illustrate Lemma 2.1 and Proposition 2.3 and also to clarify what goes wrong with the proofs of Theorems 8.2.14 and 8.2.19 in [Smi95] when the ground field does not have prime order. We then give examples illustrating Theorem 3.4.

Example 4.1. Consider the group $G \leq \mathrm{Gl}_n(\mathbb{F})$ generated by the matrices

$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where $\mathbb{F} = \mathbb{F}_p(a, b, c)$ for some nonzero a, b, c in $\mathbb{F}$. The group G fixes the hyperplane defined by the equation $z_3 = 0$. Lemma 2.1 is transparent in this example: The set $\mathcal{R}(G)$ of root vectors of G is just the $\mathbb{F}_p$-span of $(a, 0, 0), (0, b, 0)$ and $(c, c, 0)$ in $\mathbb{F}^3$. The minimum number of vectors needed to span $\mathcal{R}(G)$ is exactly the minimum number of group elements needed to generate G. Thus, G can be generated by d elements and no fewer if and only if the dimension of $\mathcal{R}(G)$ over $\mathbb{F}_p$ is d.

Assume A, B, and C form a minimum generating set for G. (In other words, either a and c are independent over $\mathbb{F}_p$ or b and c are independent over $\mathbb{F}_p$.) The group G has a polynomial ring of invariants with basic invariants

$$\begin{aligned} f_1 &= (z_1^p - a^{p-1} z_1 z_3^{p-1})^p - c^{p-1}(a^{p-1} - c^{p-1})(z_1^p - a^{p-1} z_1 z_3^{p-1}) z_3^{p(p-1)}, \\ f_2 &= (z_2^p - b^{p-1} z_2 z_3^{p-1}) - \left(\frac{b^{p-1} - c^{p-1}}{a^{p-1} - c^{p-1}} \right) (z_1^p - a^{p-1} z_1 z_3^{p-1}), \\ f_3 &= z_3. \end{aligned}$$

These basic invariants are given by the proof of Proposition 2.3.

In the special case where $a = b = 1$ and c is algebraic over $\mathbb{F}_p$ but $c \notin \mathbb{F}_p$, the group G is defined over the finite field $\mathbb{F}_p(c) = \mathbb{F}$, yet Theorem 8.2.14 and the proof of Theorem 8.2.19 in [Smi95] do not describe the group and basic invariants.

Example 4.2. Let $\mathbb{F}$ be the finite field $\mathbb{F}_q$ and let $\mathcal{A}$ be the set of all hyperplanes in $V = \mathbb{F}^n$. For each hyperplane H, choose some $l_H \in V^*$ with $\ker l_H = H$, and let Q be the product of all these linear forms: $Q = \prod_{H\in\mathcal{A}} l_H$. Then Q has degree $|\mathcal{A}| = |V|/|\mathbb{F}^*| = (q^n-1)/(q-1)$.

The group $G = \mathrm{Gl}_n(\mathbb{F})$ is generated by reflections about all hyperplanes in V. The invariants of G form a polynomial ring: $\mathbb{F}[V]^G \; = \; \mathbb{F}[f_1,\ldots,f_n]$, where f_{i+1} is the Dickson polynomial

$$d_{n,i} \;=\; \sum_{\substack{W\le V\\ \operatorname{codim} W = i}} \; \prod_{\substack{v\in V^*,\\ v|_W\neq 0}} v$$

of degree $q^n - q^i$ (for example, see [Ben93, Prop. 8.1.3]).

Note that $f_1 = d_{n,0} \;=\; Q^{q-1}$.

Fix some hyperplane H in V and let G_H be its pointwise stabilizer in G. In an appropriate basis (with $z_n = l_H$), $\mathbb{F}[V]^{G_H} = F[u_1,...,u_n]$ where

$$u_i = z_i^q - z_n^{q-1} z_i \;\text{ for }\; i<n \;\text{ and }\; u_n = z_n^{q-1}$$

(see [LS87]). Note that the sum of the exponents of G_H is

$$m_H = (n-1)(q-1) + (q-2) = n(q-1) - 1.$$

Since each f_i lies in $\mathbb{F}[V]^{G_H}$, each $f_i = h_i(u_1,\ldots,u_n)$ for some $h_i \in \mathbb{F}[V]$. Then

$$\partial f_i/\partial z_k = \sum_j (\partial h_i/\partial u_j)(\partial u_j/\partial z_k)$$

is divisible by z_n^{q-2} if $k=n$ and divisible by z_n^{q-1} otherwise. Hence, $J = \det(\partial f_i/\partial z_k)$ is divisible by $z_n^{m_H}$. As H was arbitrary, $\prod_{H\in\mathcal{A}} l_H^{m_H}$ divides J. But one may check that

$$\deg J = \deg Q\cdot(n(q-1)-1)\,,$$

and thus

$$J \;=\; Q^{n(q-1)-1} \;=\; \prod_{H\in\mathcal{A}} l_H^{m_H}.$$

Alternatively, one can use the description of the Dickson invariants in terms of Vandermonde-like determinants given in [Wil83, Prop. 1.3] to verify that $z_n^{n(q-1)-1}$ divides J:

$$d_{n,k} = \Delta_k\Delta^{-1} \text{ where } \begin{cases} \Delta_k \;=\; \det\left(z_j^{q^i}\right)_{\substack{i=0,\ldots n,\; i\neq k\\ j=1,\ldots,n}}, \\ \Delta \;\;=\; \det\left(z_j^{q^i}\right)_{\substack{i=0,\ldots n-1\\ j=1,\ldots,n}}. \end{cases}$$

Apply the quotient rule to $\partial/\partial x_i(d_{n,k})$ and expand $\partial/\partial x_i(\Delta_k)$ and $\partial/\partial x_i(\Delta)$ about the i-th row. If $i\neq n$, then z_n^{q+1} divides $\partial/\partial x_i(\Delta_k)$ and $\partial/\partial x_i(\Delta)$ and hence z_n^{q-1} divides $\partial/\partial x_i(d_{n,k})$. If $i=n$, expand Δ_k and Δ about the n-th row and cancel terms to see that z_n^{q-2} divides $\partial/\partial z_i(d_{n,j})$. The last column of this Jacobian matrix is divisible by z_n^{q-2} and the other columns are each divisible by z_n^{q-1}.

The Jacobian of the Dickson invariants was also examined by K. Kuhnigk in her Doktorarbeit ([Kuh03]).

Example 4.3. Let $G = \mathrm{Sl}_n(\mathbb{F}_q)$. As in the last example, let $\mathbb{F} = \mathbb{F}_q$ and $Q = \prod_{H\in\mathcal{A}} l_H$, where $\mathcal{A}$ is the set of all hyperplanes in $V = \mathbb{F}^n$. Every reflection in G is a transvection and

$$\mathbb{F}[V]^G \;=\; \mathbb{F}[f_1, f_2, \ldots, f_n],$$

where $f_1 = Q$ and f_{i+1} is the Dickson invariant $d_{n,i}$ (for $i \geq 1$). Fix some hyperplane H in V and let G_H be its pointwise stabilizer. In an appropriate basis (with $z_n = l_H$), $\mathbb{F}[V]^{G_H} = F[u_1, ..., u_n]$ where $u_i = z_i^q - z_n^{q-1} z_i$ for $i < n$ and $u_n = z_n$. Note that the sum of the exponents of G_H is $m_H = (n-1)(q-1)$. Since each f_i lies in $\mathbb{F}[V]^{G_H}$, $f_i = h_i(u_1, \ldots, u_n)$ for some $h_i \in \mathbb{F}[V]$. Then

$$\partial f_i / \partial z_k = \sum_j (\partial h_i / \partial z_j)(\partial u_j / \partial z_k)$$

is divisible by z_n^{q-1} if $k \neq n$. Hence, $J = \det(\partial f_i / \partial z_k)$ is divisible by $z_n^{(n-1)(q-1)}$ and thus $Q^{\, m_H} = \prod_{H \in \mathcal{A}} l_H^{m_H}$ divides J. But

$$\deg J = (q^n - 1)(n-1) = \deg Q (n-1)(q-1),$$

and so

$$J \;=\; Q^{(n-1)(q-1)} \;=\; \prod_{H \in \mathcal{A}} l_H^{\, m_H}.$$

Note that for both groups $\mathrm{Gl}_n(\mathbb{F}_q)$ and $\mathrm{Sl}_n(\mathbb{F}_q)$, the Jacobian determinant J lies in the Hilbert ideal generated by the basic invariants: The image of J in the coinvariant algebra $\mathbb{F}[V]/(f_1, \ldots, f_n)$ is zero in both cases.

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