A Sharp Ratio Inequality for Optimal Stopping When Only Record Times are Observed

Pieter C. Allaart
Department of Mathematics, University of North Texas, Denton, Texas, USA

Abstract: Let $X_1, \ldots, X_n$ be independent, identically distributed random variables that are non-negative and integrable, with known continuous distribution. These random variables are observed sequentially, and the goal is to maximize the expected $X$ value at which one stops. Let $V_n$ denote the optimal expected return of a player who can observe at time $j$ only whether $X_j$ is a relative record ($j = 1, \ldots, n$), and $W_n$ that of a player who observes at time $j$ the actual value of $X_j$. It is shown that $V_n > a_n W_n$, where $a_n = \max_{1\leq k<n} (k/n) \sum_{j=k}^{n-1} 1/j$, and this inequality is sharp.

Keywords: Optimal stopping rule; Relative record; Secretary problem.

Subject Classifications: 60G40.

Let $X, X_1, \ldots, X_n$ be independent, identically distributed (i.i.d.) random variables that are nonnegative and integrable, with known continuous distribution. Recently, Samuel-Cahn (2007) investigated the problem of maximizing $EX_\tau$ over those stopping times $\tau$ that use only information about the record times of the sequence $X_1, \ldots, X_n$. Precisely, let $Y_1 \equiv 1$, and for $j = 2, \ldots, n$, let $Y_j = 1$ if $X_j > \max\{X_1, \ldots, X_{j-1}\}$, and $Y_j = 0$ otherwise. Define the $\sigma$-algebras $\mathcal{G}_j := \sigma(\{Y_1, \ldots, Y_j\})$, $j = 1, \ldots, n$, and let

$$V_n := \sup_{\tau \leq n} EX_\tau,$$

the supremum being over all stopping times $\tau$ adapted to the filtration $\{\mathcal{G}_j\}$ with $\tau \leq n$ a.s.

Samuel-Cahn shows that the optimal rule is among the rules

$$t_n(k) = \min \{j > k : Y_j = 1\} \wedge n, \quad k = 1, \ldots, n-1,$$

and the expected payoff from the rule $t_n(k)$ is

$$V_n(k) := EX_{t_n(k)} = k \left[ \sum_{j=k+1}^{n-1} \frac{EM_j}{j(j-1)} + \frac{EX}{n-1} \right], \quad (1)$$

where $M_j := \max\{X_1, \ldots, X_j\}$. Thus, $V_n = \max_{1\leq k<n} V_n(k)$.

Samuel-Cahn goes on to describe the asymptotic properties of $V_n$ as $n \to \infty$ for various distributions of $X$, and to give asymptotic comparisons of $V_n$ with the usual optimal stopping value

$$W_n := \sup_{\tau \leq n} EX_\tau.$$
where the supremum is over all stopping times \( \tau \leq n \) adapted to the filtration \( \{ \mathcal{F}_j \} \) defined by \( \mathcal{F}_j := \sigma(\{X_1, \ldots, X_j\}) \), \( j = 1, \ldots, n \).

For fixed \( n \), however, (1) immediately yields a sharp inequality comparing \( V_n \) and \( W_n \). Let

\[
s(k, n) := \sum_{j=k}^{n} \frac{1}{j}, \quad a_n := \max_{1 \leq k < n} (k/n)s(k, n - 1).
\]

**Theorem 1.** For every \( n \geq 2 \),

\[
V_n > a_n W_n,
\]

and the constant \( a_n \) is best possible.

**Proof.** Let \( M_0 \equiv 0 \). Since \( X \) is continuous it is not degenerate, and therefore (as is easy to show), \( EM_{j+1} - EM_j < EM_j - EM_{j-1} \) for all \( j \geq 1 \). Hence, there is a strictly concave function that interpolates the points \( (j, EM_j) \), \( j = 0, 1, \ldots, n \), so that

\[
EM_j > \frac{j}{n} EM_n, \quad j = 1, \ldots, n - 1.
\]

Substituting this into (1) yields

\[
V_n > \max_{1 \leq k < n} k \left[ \sum_{j=k+1}^{n-1} \frac{EM_n}{n(j-1)} + \frac{EM_n}{n(n-1)} \right] = a_n EM_n \geq a_n W_n.
\]

To see that the bound is sharp, let \( X \) initially have distribution

\[
P(X = K) = K^{-1} = 1 - P(X = 0),
\]

where \( K > 1 \). Note that for this distribution, \( W_n = EM_n \). Moreover,

\[
EM_j = K P(M_j = K) = K[1 - (1 - K^{-1})^j] \rightarrow j \quad \text{as} \quad K \rightarrow \infty,
\]

and so, given \( \varepsilon > 0 \), \( K \) may be chosen so large that

\[
EM_j \leq \left( \frac{j}{n} + \varepsilon \right) EM_n
\]

for \( j = 1, \ldots, n \). Hence

\[
V_n(k) \leq a_n + k \sum_{j=k+1}^{n} \frac{\varepsilon}{j(j-1)} EM_n
\]

\[
= a_n + \left( 1 - \frac{k}{n} \right) \varepsilon EM_n < (a_n + \varepsilon) EM_n.
\]

Of course, the distribution in (3) is not continuous, but it may be approximated arbitrarily closely by a continuous distribution with respect to for instance the Lévy-Prohorov metric.
Since the functionals $E_{Mj}$ are continuous with respect to this metric, the sharpness of the bound (2) follows. □

It is interesting to note that the constant $a_n$ is exactly the probability of selecting the best of $n$ items under an optimal strategy in the classical secretary problem or dowry problem. Gilbert and Mosteller (1966, pp. 39-41) gave a large table of values of $a_n$ (for instance, $a_2 = a_3 = 1/2$, $a_4 = 11/24$, $a_5 = 13/30$, etc.) and showed that $\lim_{n \to \infty} a_n = 1/e$, a fact that has since become common knowledge. However, it does not appear to have been demonstrated explicitly in the literature that the $a_n$'s are strictly decreasing from $a_3$ onward. This intuitively plausible fact depends on the following elegant result from number theory due to Kürschák (1918); see also Pólya and Szegő (1976, Problem VIII.251, p. 154 and pp. 358/59).

**Lemma 1.** Let $k$ and $n$ be positive integers with $k < n$. Then $s(k,n)$ is not an integer.

**Proposition 1.** The constants $\{a_n\}$ are strictly decreasing for $n \geq 3$.

**Proof** Let

$$\alpha_{n,k} := (k/n)s(k,n-1), \quad k = 1, \ldots, n-1,$$

and let $k_n$ be the value of $k$ that maximizes $\alpha_{n,k}$ for fixed $n$. It follows as in Gilbert and Mosteller (1966, p. 39) that

$$k_n = \min \{k : s(k+1,n-1) < 1\}. \quad (4)$$

From this it is clear that $k_{n+1}$ is equal to either $k_n$ or $k_n + 1$. If $k_{n+1} = k_n$, then

$$a_n - a_{n+1} = \alpha_{n,k_n} - \alpha_{n+1,k_n}$$

$$= k_n \left[ \frac{s(k_n,n-1)}{n} - \frac{s(k_n,n)}{n+1} \right]$$

$$= k_n \frac{n(n+1)}{n(n+1)} (s(k_n,n-1) - 1) > 0$$

by (4), the inequality being strict for $n \geq 3$ in view of Lemma 1. And if $k_{n+1} = k_n + 1$, then

$$a_n - a_{n+1} = \alpha_{n,k_n} - \alpha_{n+1,k_n+1}$$

$$= \frac{k_n}{n} s(k_n,n-1) - \frac{k_n + 1}{n+1} s(k_n+1,n)$$

$$= \frac{n - k_n}{n(n+1)} (1 - s(k_n+1,n-1)) > 0$$

by (4). Thus, $a_n$ is strictly decreasing for $n \geq 3$. □

**Remarks.** (a) It is well known that $k_n \approx n/e$, the approximation being fairly accurate even for small values of $n$. Thus, the rule “let roughly a proportion $1/e$ of the observations go by, then stop with the next relative record” guarantees an expected payoff at least $a_n$.
times the payoff of a player who can observe the actual $X_i$'s, regardless of the distribution of $X$.

(b) It is clear from the proof of Theorem 1 that even a “prophet” who can foretell the future can do no better than $1/a_n$ times the value of a player observing only record times. In particular, the advantage of a prophet over such a player is bounded by the factor $e$.

(c) Hill and Kertz (1982) showed that for i.i.d. random variables, $E M_n \leq c_n W_n$, where $\{c_n\}$ is a sequence of numbers, believed to be strictly increasing, with known limit $1.34149 \cdots$. Seen in this light, the result of this note shows that the opportunity to fully observe the random variables gives a much greater advantage than the gift of foresight.

REFERENCES


