1. **Finite sums and products.**
   Prove using the mathematical induction principle that for every positive integer $n$:
   
   a) \[
   \sum_{k=1}^{n} k^3 = \frac{1}{6}n(n+1)(2n+1)
   \]
   
   b) \[
   \sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2
   \]
   
   c) \[
   \sum_{k=1}^{n} (k+1)2^k = n2^{n+1}
   \]
   
   d) \[
   \sum_{k=1}^{n} \ln \left(1 + \frac{1}{k}\right) = \ln (1+n)
   \]
   
   e) \[
   \sum_{k=0}^{n} \frac{(k+3)!}{k!} = \frac{1}{4} \frac{(n+4)!}{n!}
   \]
   
   f) \[
   \sum_{k=2}^{2n+1} \frac{1}{k(k+1)} = \frac{n}{2(n+1)}
   \]
   
   g) \[
   \sum_{k=2}^{2n+1} (3k - 1) = n(6n + 7)
   \]
   
   h) \[
   \sum_{k=n+1}^{2n} 2k = 3n^2 + n
   \]
   
   i) \[
   \sum_{k=1}^{2n} \frac{(-1)^k}{k} = \sum_{k=n+1}^{2n} \frac{1}{k}
   \]
   
   j) \[
   \prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \frac{n+1}{2n}
   \]
   
   k) \[
   \prod_{k=1}^{n} (2k - 1) = \frac{(2n)!}{2^n n!}
   \]

2. **This example shows that the first step is essential!**
   Consider the statement \[ \sum_{k=1}^{n} k = \frac{1}{8}(2n+1)^2 \] for every positive integer $n$.
   
   a) Show that if the statement holds for some positive integer $n = k$, then it also holds for $n = k + 1$. (In other words, give a proof by induction without Step 1).
   
   b) Is the statement true?

3. **Divisibility.**
   Prove by induction that for every positive integer $n$:
   
   a) \[ 3^{2n+1} + 2^{n-1} \] is divisible by 7
   
   b) \[ 5 \cdot 3^{4n+1} - 2^n \] is divisible by 7
   
   c) \[ 3^{2^n} - 1 \] is divisible by \( 2^{n+2} \)
   
   d) \[ 3^{2^n} - 1 \] is not divisible by \( 2^{n+3} \)
   
   e) \[ x^n - y^n \] is divisible by \( x - y \), \( (x, y \in \mathbb{R}, x \neq y) \)
   
   f) \[ x^{2n} - y^{2^n} \] is divisible by \( x + y \), \( (x, y \in \mathbb{R}, x \neq -y) \)
4. Inequalities.

a) Prove (by induction, if necessary):
   
   (i) \( n + 1 \leq 2^n \leq (n + 1)! \quad [\text{all } n] \)
   
   (ii) \( n^2 \leq 2^n \leq n! \quad [n \geq 4] \)
   
   (iii) \( 2^{n+1} > n^2 + n + 2 \quad [n \geq 3] \)
   
   (iv) \( n! \leq n^n \leq (n!)^2 \quad [\text{all } n] \)

b) Prove that for \( h \geq 2 \) and \( n \geq 2(\text{an integer}) \):

\[
(1 + h)^n \geq \frac{1}{2} n(n - 1)h^2
\]

What can you prove when just \( h \geq 0 \) is given?

5. Higher order derivatives.

a) Let \( f(x) = \sin x \). Prove by induction that

\[
f^{(n)}(x) = \sin \left(x + \frac{1}{2} n\pi\right)
\]

b) For every \( n \), determine the \( n^{th} \) order derivative of \( f_k(x) \), \( (k = 1, 2, 3) \).

Prove your formula by induction.

(i) \( f_1(x) = \cos x \)

(ii) \( f_2(x) = \frac{1}{1 + x} \)

(iii) \( f_3(x) = \ln x \)

c) Prove: \( \frac{d^n}{dx^n} f(ax) = a^n f^{(n)}(ax) \).

Now use a) to find a formula for \( \frac{d}{dx}(\sin 2x) \)

d) The function \( f : (-1, 1) \to \mathbb{R} \) is given by

\[
f(x) = \frac{2}{x^2 - 1}.
\]

Prove using the induction principle, that

\[
f^{(n)}(x) = (-1)^n n! \left\{ (x - 1)^{-n-1} - (x + 1)^{-n-1}\right\}
\]