Math 4610, Review for Exam 3, Solutions

1. a) $\left(\frac{9}{10}\right)^5$ b) $\frac{1}{1/10} = 10$ c) $20 \cdot \frac{1}{10} = 2$ d) $\binom{19}{2}\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^{17}$

2. Let T be the atom's lifetime and assume $T \sim Exp(\lambda)$. Then $\lambda = \frac{\ln 2}{1000}$ and so $P(T>t) = e^{-\lambda t} = \left(e^{\ln 2}\right)^{-t/1000} = 2^{-t/1000}$. Thus, using the memoryless property,
$P(T \le 1500 \mid T > 1000) = 1 - P(T>1500 \mid T>1000) = 1 - P(T>500) = 1 - 2^{-500/1000} = 1 - 2^{-1/2}$
$= \boxed{.2929}$

3. $E(XY) = E(X)E(Y) = \mu^2$, and $E[(XY)^2] = E[X^2Y^2] = E(X^2)E(Y^2)$
$= \{Var(X) + (E(X))^2\} \cdot \{Var(Y) + (E(Y))^2\} = (\sigma^2+\mu^2)^2$, so $Var(XY) = E[(XY)^2] - [E(XY)]^2$
$= (\sigma^2+\mu^2)^2 - (\mu^2)^2 = (\sigma^4 + 2\sigma^2\mu^2 + \mu^4) - \mu^4 = \sigma^4 + 2\sigma^2\mu^2 = \sigma^2(\sigma^2+2\mu^2)$.

4. a) $\int_0^1 x(1-x)^2dx = \int_0^1 (x-2x^2+x^3)dx = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12} \Rightarrow \boxed{c=12}$
b) $P(X \le \frac{1}{2}) = \int_0^{1/2} 12x(1-x)^2dx = \left[6x^2 - 8x^3 + 3x^4\right]_0^{1/2} = \frac{3}{2} - 1 + \frac{3}{16} = \boxed{\frac{11}{16}}$
c) $P(X = \frac{1}{3}) = 0$.
d) $E(X) = \int_0^1 12x^2(1-x)^2dx = \left[4x^3 - 6x^4 + \frac{12}{5}x^5\right]_0^1 = 4 - 6 + \frac{12}{5} = \boxed{\frac{2}{5}}$.
e) $E(X^2) = \int_0^1 12x^3(1-x)^2dx = \int_0^1 (12x^3 - 24x^4 + 12x^5)dx = \frac{12}{4} - \frac{24}{5} + \frac{12}{6} = \frac{1}{5}$
$\therefore Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{5} - \frac{4}{25} = \boxed{\frac{1}{25}}$.

5. Assume the weights of the hotel guests are independent of each other and all have the same distribution. Let X_i = weight of the i^{th} guest in the elevator, $i = 1, \dots, 30$, and let $S_{30} := X_1 + \dots + X_{30}$ be the total weight. Then $E(S_{30}) = (30)(150) = 4500$ and $SD(S_{30}) = \sqrt{30}(55) = 301.25$. By the Central Limit Theorem,
$$P(S_{30} > 5000) \approx 1 - \Phi\left(\frac{5000-4500}{301.25}\right) = 1 - \Phi(1.66) = 1 - .9515 = \boxed{.0485}$$
Note: Continuity correction makes little sense here (why?), but also does little harm.

6. a) Let X be the number of brown chocolate chips in this cookie (not "per cookie", "in a cookie")
It's reasonable to assume $X \sim Ps(4)$, so $P(X \le 2) = e^{-4}(1+4+\frac{4^2}{2}) = 13e^{-4} \approx \boxed{.2381}$
b) Let Y be the number of white chocolate chips in the cookie; assume $Y \sim Ps(2)$ and Y is independent of X. Let $Z := X+Y$ be the total number of chocolate chips in the cookie.
Then $Z \sim Ps(6)$, so $P(Z \le 5) = e^{-6}\left(1 + 6 + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!} + \frac{6^5}{5!}\right) \approx \boxed{.4457}$
c) Under the same assumptions as before, $P(X=3, Y=2 \mid Z=5) = \frac{P(X=3,Y=2)}{P(Z=5)}$
$$= \frac{e^{-2}\cdot\frac{2^2}{2!}\cdot e^{-4}\cdot\frac{4^3}{3!}}{e^{-6}\cdot\frac{6^5}{5!}} = \frac{5!}{2!3!}\cdot\frac{2^2\cdot 4^3}{6^5} = \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 \approx \boxed{0.3292}$$

7. a) $N = I_A + I_B$; b) $E(N) = E(I_A) + E(I_B) = P(A) + P(B) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$.

c) $Var(N) = Var(I_A) + Var(I_B)$ [because I_A, I_B independent!] $= \frac{1}{3}(1-\frac{1}{3}) + \frac{1}{2}(1-\frac{1}{2}) = \frac{17}{36}$

d) Since $A \subset B$, if A occurs, then B occurs also. Thus, $P(N=0) = P(B^c) = \frac{1}{2}$,

$P(N=1) = P(B \text{ but not } A) = P(B) - P(A) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, $P(N=2) = P(A \text{ and } B) = P(A) = \frac{1}{3}$.

So $E(N^2) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{3} = \frac{1}{6} + \frac{4}{3} = \frac{3}{2}$, and $Var(N) = \frac{3}{2} - (\frac{5}{6})^2 = \frac{54}{36} - \frac{25}{36} = \frac{29}{36}$.

8. a) $P(A_n) = P(\text{first } n-1 \text{ draws, then } A \text{ wins}) = p_D^{n-1} p_A$.

b) $P(A \text{ wins overall}) = \sum_{n=1}^{\infty} P(A \text{ wins overall in } n \text{ games}) = \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} p_D^{n-1} p_A$

$= p_A \cdot \frac{1}{1-p_D} = \frac{p_A}{p_A + p_B}$.

9. Let T be your total waiting time; then $T \sim \exp(\lambda)$ with $\lambda = \frac{1}{10}$.
The required probability is

$$P(T \le 12 \mid T > 10) = 1 - P(T > 12 \mid T > 10) = 1 - P(T > 2) \quad \text{(memoryless property!)}$$
$$= 1 - e^{-\lambda \cdot 2} = 1 - e^{-1/5} \approx .1813.$$

10. Let N = total # of misprints, X = # of misprints found. Then $P(X=k) = \sum_{n=k}^{\infty} P(X=k \mid N=n) P(N=n)$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} e^{-\mu} \frac{\mu^n}{n!} = e^{-\mu} \sum_{n=k}^{\infty} \frac{p^k (1-p)^{n-k} \mu^n}{k!(n-k)!} = e^{-\mu} \cdot \frac{p^k \mu^k}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} \mu^{n-k}}{(n-k)!}$$
$$= e^{-\mu} \frac{(\mu p)^k}{k!} \sum_{\ell=0}^{\infty} \frac{[(1-p)\mu]^\ell}{\ell!} = e^{-\mu} \frac{(\mu p)^k}{k!} e^{(1-p)\mu} = e^{-\mu p} \frac{(\mu p)^k}{k!} \quad \therefore \boxed{X \sim Ps(\mu p)}$$