Math 4610 Exam 2 - Solutions

1. Sampling without replacement!

a) There are $\binom{52}{13}$ total possible hands, all equally likely. There are $\binom{4}{2}$ ways to chose 2 aces out of 4, and $\binom{48}{11}$ ways to choose 11 non-aces out of 48. Thus,

$$P(\text{two aces}) = \binom{4}{2}\binom{48}{11}/\binom{52}{13}.$$

b) There are $\binom{4}{2}$ ways to chose 2 aces, $\binom{4}{3}$ ways to chose 3 kings, and $\binom{44}{8}$ ways to chose 8 other cards. Thus, $P(\text{two aces, three kings}) = \binom{4}{2}\binom{4}{3}\binom{44}{8}/\binom{52}{13}$.

2. Each roll is an independent trial with 6 possible outcomes $(1,2,\dots,6)$, each of which has probability $p_i = \frac{1}{6}$ $(i=1,\dots,6)$. Let $N_1 = \#$ of ones, $N_2 = \#$ of twos, ..., $N_6 = \#$ of sixes. Then $(N_1,\dots,N_6) \sim$ multinomial $(12; \frac{1}{6},\dots,\frac{1}{6})$, and the required probability is

$$P(N_1=2, N_2=2, \dots, N_6=2) = \frac{12!}{2!2!\cdots2!}\cdot\left(\tfrac{1}{6}\right)^2\left(\tfrac{1}{6}\right)^2\cdots\left(\tfrac{1}{6}\right)^2 = \frac{12!}{2^6}\left(\tfrac{1}{6}\right)^{12}.$$

3. Let X be the number of times 6 appears in the seven rolls. Then $X \sim$ binomial $(7, \frac{1}{6})$.

a) $P(6 \text{ appears at most once}) = P(X \le 1) = P(X=0)+P(X=1) = \left(\frac{5}{6}\right)^7 + \binom{7}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^6.$

b) $$P(\text{two 6's in first four} \mid \text{three 6's in the seven}) = \frac{P(\text{two 6's in first four } \underline{\text{and}} \text{ three in the seven})}{P(\text{three 6's in the seven})}$$

$$= \frac{P(\text{two 6's in first four } \underline{\text{and}} \text{ one 6 in last three})}{P(\text{three 6's in the seven})} = \frac{P(\text{two 6's in four})P(\text{one 6 in three})}{P(\text{three 6's in seven})}$$

First four rolls are independent of the last three.

$$= \frac{\binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2\cdot\binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2}{\binom{7}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^4} = \frac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}}.$$

4. a)

$X_2 \backslash X_1$	1	2	3	4
1	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	1/16
4	1/16	1/16	1/16	1/16

b)

$Y_2 \backslash Y_1$	1	2	3	4
1	1/16	0	0	0
2	1/8	1/16	0	0
3	1/8	1/8	1/16	0
4	1/8	1/8	1/8	1/16

[e.g. $P(Y_1=1, Y_2=2) =$
$= P(X_1=1, X_2=2) + P(X_1=2, X_2=1)$
$= \frac{1}{16}+\frac{1}{16} = \frac{1}{8}$, etc.]

c) Range of D: $\{0,1,2,3\}$.

d) $P(D=0) = P(Y_2-Y_1=0) = P(Y_1=1,Y_2=1)+P(Y_1=2,Y_2=2)+P(Y_1=3,Y_2=3)+P(Y_1=4,Y_2=4) = \frac{4}{16} = \frac{1}{4}.$

$P(D=1) = P(Y_2-Y_1=1) = P(Y_1=1,Y_2=2)+P(Y_1=2,Y_2=3)+P(Y_1=3,Y_2=4) = \frac{3}{8}.$

$P(D=2) = P(Y_2-Y_1=2) = P(Y_1=1,Y_2=3)+P(Y_1=2,Y_2=4) = \frac{2}{8} = \frac{1}{4}.$

$P(D=3) = P(Y_2-Y_1=3) = P(Y_1=1,Y_2=4) = \frac{1}{8}.$

e) $E(D) = 0\cdot P(D=0)+1\cdot P(D=1)+2\cdot P(D=2)+3\cdot P(D=3)$

$= 0\cdot\frac{1}{4}+1\cdot\frac{3}{8}+2\cdot\frac{1}{4}+3\cdot\frac{1}{8} = \frac{5}{4}.$

5. There are $\binom{10}{3} = 120$ ways to choose 3 people from 10.

a) In 8 of these do the 3 people stand next to each other (one combination for each possible position of the leftmost person in the group of three). Thus, the required probability is $8/120 = \frac{1}{15}$.

b) Here there are 10 combinations in which the 3 people stand next to each other: one for each position on the circle to start the line of 3. So the required probability is $10/120 = \frac{1}{12}$.

6. a) The draws are independent trials, but with different success probabilities. Hence, X does <u>not</u> have a binomial distribution.

b) Let $A_i = \{\text{ball from } i^{th} \text{ urn is red}\}$, $i = 1,2,3$. Then $X = I_{A_1} + I_{A_2} + I_{A_3}$, so

$$E(X) = E(I_{A_1}) + E(I_{A_2}) + E(I_{A_3}) = P(A_1) + P(A_2) + P(A_3) = \frac{1}{3} + \frac{3}{4} + \frac{1}{2} = \frac{19}{12}.$$

7. Let X be the number of working devices in a box chosen at random. Then $X \sim \text{binomial}(n,p)$ with $n = 50$ and $p = 0.9$. Let $\mu = np = 45$, and $\sigma = \sqrt{npq} \doteq 2.12$. We want the largest k so that $P(X \geq k) \geq 0.9$. Using the normal approximation,

$$P(X \geq k) \approx 1 - \Phi\left(\frac{k - \frac{1}{2} - \mu}{\sigma}\right) = 1 - \Phi\left(\frac{k - 45.5}{2.12}\right) = \Phi\left(\frac{45.5 - k}{2.12}\right)$$

From the normal table, this last expression is ≥ 0.9 if and only if

$$\frac{45.5 - k}{2.12} \geq 1.29 \Leftrightarrow 45.5 - k \geq 2.73 \Leftrightarrow k \leq 42.76$$

So the largest <u>integer</u> k that satisfies the inequality is $\boxed{k = 42}$

8. a) There are $\binom{100}{3}$ ways to choose the 3 winning tickets. There are 10 ways to choose a lucky person, and $\binom{10}{3}$ ways to choose 3 winning tickets from this person's 10 tickets. Thus.

$$P(\text{one person has all 3 winning tickets}) = \frac{10\binom{10}{3}}{\binom{100}{3}}.$$

b) Similarly, 10 ways to choose the person with 2 winning tickets, $\binom{10}{2}$ ways to choose the 2 winning tickets from this person's collection, and 90 ways to choose the third winning ticket from the combined holdings of the 9 other people. The required probability is

$$\frac{10\binom{10}{2} \cdot 90}{\binom{100}{3}}.$$

[Alternative: a) $\frac{10\binom{3}{3}\binom{97}{7}}{\binom{100}{10}}$; b) $\frac{10\binom{3}{2}\binom{97}{8}}{\binom{100}{10}}$. here considering the distribution of the tickets random, with the winning tickets fixed beforehand.