

Math 4610/5810, Exam 1 - Solutions

1. a) $\Omega = \{ (i,j) : i,j \in \mathbb{N}, i=1,\dots,6, j=1,\dots,6 \}$
 $= \{ (1,1), (1,2), \dots, (1,6) \}$
 $\quad (2,1), (2,2), \dots, (2,6) \}$
 $\quad \vdots$
 $\quad (6,1), (6,2), \dots, (6,6) \}$

b) $A = \{ (1,3), (3,1), (2,4), (4,2), (3,5), (5,3), (4,6), (6,4) \}$, $\#A = 8 \Rightarrow P(A) = \frac{\#A}{\#\Omega} = \frac{8}{36} = \frac{2}{9}$.

c) $B = \{ (2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6) \}$, $\#B = 9 \Rightarrow P(B) = \frac{9}{36} = \frac{1}{4}$.

d) $AB = \{ (2,4), (4,2), (4,6), (6,4) \}$, $\#(AB) = 4 \Rightarrow P(AB) = \frac{4}{36} = \frac{1}{9}$.

$\therefore P(A|B) = \frac{P(AB)}{P(B)} = \frac{1/9}{1/4} = \frac{4}{9}$.

2. a) $P(\text{exactly one}) = P(AB^cC^c \cup A^cBC^c \cup A^cB^cC) = P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC)$
 $= P(A)P(B^c)P(C^c) + P(A^c)P(B)P(C^c) + P(A^c)P(B^c)P(C)$ [A, B, C independent]
 $= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{5} = \frac{8}{30} + \frac{4}{30} + \frac{2}{30} = \frac{14}{30} = \frac{7}{15}$.

b) $P(\text{at least one}) = P(A \cup B \cup C) = 1 - P((A \cup B \cup C)^c) = 1 - P(A^cB^cC^c) =$
 $= 1 - P(A^c)P(B^c)P(C^c) = 1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 1 - \frac{4}{15} = \frac{11}{15}$.

c) $P(A | \text{one}) = \frac{P(A \cap \{\text{exactly one}\})}{P(\{\text{exactly one}\})} = \frac{P(AB^cC^c)}{P(\text{exactly one})} = \frac{\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5}}{\frac{7}{15}} = \frac{4}{7}$.

3. Let $H_i = \{ \text{heads in } i^{\text{th}} \text{ toss} \}$ and $T_i = \{ \text{tails in } i^{\text{th}} \text{ toss} \}$, $i = 1, 2, \dots$.

Then $P_1 = 0$ (since at least two tosses are needed for two heads),

$P_2 = P(H_1, H_2) = P(H_1)P(H_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$,

$P_3 = P(T_1, H_2, H_3) = P(T_1)P(H_2)P(H_3) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$,

$P_4 = P(T_1, T_2, H_3, H_4) + P(H_1, T_2, H_3, H_4) = \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = \frac{1}{8}$

$P_5 = P(T_1, T_2, T_3, H_4, H_5) + P(T_1, H_2, T_3, H_4, H_5) + P(H_1, T_2, T_3, H_4, H_5) = 3 \cdot \left(\frac{1}{2}\right)^5 = \frac{3}{32}$.

[In fact, for all r , $P_r = F_{r-1}/2^r$, where F_n denotes the n^{th} Fibonacci number!]

4. If B_1, \dots, B_n is a partition of B , then AB_1, \dots, AB_n is a partition of AB , so

$$P(AB) = P(AB_1) + P(AB_2) + \dots + P(AB_n)$$

$$= P(A|B_1)P(B_1) + \dots + P(A|B_n)P(B_n).$$

Divide by $P(B)$, and note that for $i=1, \dots, n$, $B_i \subseteq B$ and hence $B_i = B_i B$.
 Thus,

$$P(A|B) = \frac{P(AB)}{P(B)} = P(A|B_1) \frac{P(B_1)}{P(B)} + \dots + P(A|B_n) \frac{P(B_n)}{P(B)}$$

$$= P(A|B_1) \frac{P(B_1|B)}{P(B)} + \dots + P(A|B_n) \frac{P(B_n|B)}{P(B)}$$

$$= P(A|B_1)P(B_1|B) + \dots + P(A|B_n)P(B_n|B).$$

5. Let $F = \{\text{chosen die is fair}\}$, $B = \{\text{chosen die is biased}\}$, $O = \{\text{outcome is 1}\}$.
 Then $P(F) = \frac{\# \text{ fair die}}{\text{total \# dice}} = \frac{f}{f+b}$, $P(B) = \frac{b}{f+b}$, $P(O|F) = \frac{1}{6}$, $P(O|B) = \frac{1}{3}$.

So

$$P(O) = P(O|F)P(F) + P(O|B)P(B) = \frac{1}{6} \cdot \frac{f}{f+b} + \frac{1}{3} \cdot \frac{b}{f+b}$$

and

$$P(F|O) = \frac{P(F|O)P(O)}{P(O)} = \frac{\frac{1}{6} \cdot \frac{f}{f+b}}{\frac{1}{6} \cdot \frac{f}{f+b} + \frac{1}{3} \cdot \frac{b}{f+b}} = \frac{f}{f+2b}$$

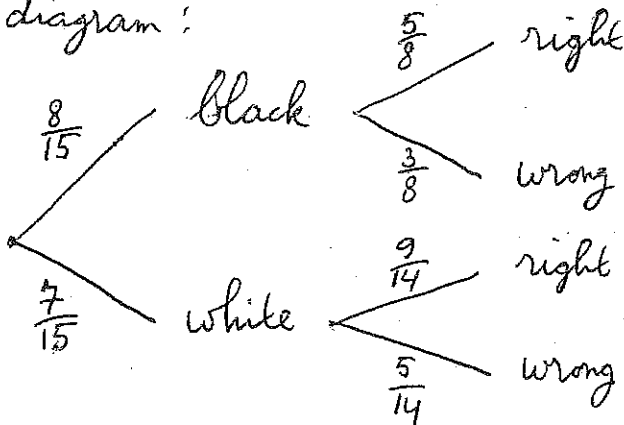
6. a) $P(\text{white}) = P(\text{white}|\text{Box 1})P(\text{Box 1}) + P(\text{white}|\text{Box 2})P(\text{Box 2})$
 $= \frac{1}{3} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{2} = \frac{1}{6} + \frac{3}{10} = \frac{5}{30} + \frac{9}{30} = \frac{14}{30} = \frac{7}{15}$ (again!)

b) $P(\text{Box 1}|\text{white}) = \frac{P(\text{white}|\text{Box 1})P(\text{Box 1})}{P(\text{white})} = \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{7}{15}} = \frac{1}{6} \cdot \frac{15}{7} = \frac{5}{14}$

c) $P(\text{Box 1}|\text{black}) = \frac{P(\text{black}|\text{Box 1})P(\text{Box 1})}{P(\text{black})} = \frac{\frac{2}{3} \cdot \frac{1}{2}}{1 - \frac{7}{15}} = \frac{1/3}{8/15} = \frac{5}{8}$

d) If the ball is white, guess box 2 (correct with probability $\frac{9}{14}$)
 and if black, guess box 1 (correct with probability $\frac{5}{8}$).

Tree diagram:



e) $P(\text{guess correct}) = P(\text{guess correct}|\text{black})P(\text{black}) + P(\text{guess correct}|\text{white})P(\text{white})$
 $= \frac{5}{8} \cdot \frac{8}{15} + \frac{9}{14} \cdot \frac{7}{15}$
 $= \frac{1}{3} + \frac{9}{30} = \frac{19}{30}$

7. Let X be the number of times 6 appears in the seven rolls. Then $X \sim \text{binomial}(7, \frac{1}{6})$.

a) $P(6 \text{ appears at most once}) = P(X \leq 1) = P(X=0) + P(X=1) = \left(\frac{5}{6}\right)^7 + \binom{7}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^6$

b) $P(\text{two 6's in first four} | \text{three 6's in the seven}) = \frac{P(\text{two 6's in first four and three in the seven})}{P(\text{three 6's in the seven})}$

$$= \frac{P(\text{two 6's in first four and one 6 in last three})}{P(\text{three 6's in the seven})} = \frac{P(\text{two 6's in four})P(\text{one 6 in three})}{P(\text{three 6's in seven})}$$

$$= \frac{\binom{4}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 \cdot \binom{3}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2}{\binom{7}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^4} = \frac{\binom{4}{2} \binom{3}{1}}{\binom{7}{3}}$$

First four rolls are independent of the last three.