Math 4610, Review for Exam 1 (Spring 2016), Solutions

1. a) $\Omega = \{ (i,j) : i,j \in \mathbb{N},\ i=1,\dots,6,\ j=1,\dots,6 \}$

$$= \{ (1,1), (1,2), \dots, (1,6)$$
$$(2,1), (2,2), \dots, (2,6)$$
$$\vdots$$
$$(6,1), (6,2), \dots, (6,6) \}$$

b) $A = \{(1,3), (3,1), (2,4), (4,2), (3,5), (5,3), (4,6), (6,4)\}$, $\#A = 8 \Rightarrow P(A) = \frac{\#A}{\#\Omega} = \frac{8}{36} = \frac{2}{9}$.

c) $B = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\}$, $\# B = 9 \Rightarrow P(B) = \frac{9}{36} = \frac{1}{4}$.

d) $AB = \{(2,4), (4,2), (4,6), (6,4)\}$, $\#(AB) = 4 \Rightarrow P(AB) = \frac{4}{36} = \frac{1}{9}$.

$\therefore P(A|B) = \frac{P(AB)}{P(B)} = \frac{1/9}{1/4} = \frac{4}{9}$.

2. a) $P(\text{exactly one}) = P(AB^cC^c \cup A^cBC^c \cup A^cB^cC) = P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC)$

$= P(A)P(B^c)P(C^c) + P(A^c)P(B)P(C^c) + P(A^c)P(B^c)P(C)$ [A, B, C independent]

$= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{5} = \frac{8}{30} + \frac{4}{30} + \frac{2}{30} = \frac{14}{30} = \frac{7}{15}$.

b) $P(\text{at least one}) = P(A\cup B\cup C) = 1 - P((A\cup B\cup C)^c) = 1 - P(A^cB^cC^c) =$

$= 1 - P(A^c)P(B^c)P(C^c) = 1 - \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5} = 1 - \frac{4}{15} = \frac{11}{15}$.

c) $P(A \mid \text{exactly one}) = \frac{P(A \cap \{\text{exactly one}\})}{P(\text{exactly one})} = \frac{P(AB^cC^c)}{P(\text{exactly one})} = \frac{\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}}{\frac{7}{15}} = \frac{4}{7}$.

3. Let $H_i = \{\text{heads in } i^{th} \text{ toss}\}$ and $T_i = \{\text{tails in } i^{th} \text{ toss}\}$, $i = 1, 2, \dots$.
Then $P_1 = 0$ (since at least two tosses are needed for two heads),

$P_2 = P(H_1H_2) = P(H_1)P(H_2) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$,

$P_3 = P(T_1H_2H_3) = P(T_1)P(H_2)P(H_3) = (\frac{1}{2})^3 = \frac{1}{8}$,

$P_4 = P(T_1T_2H_3H_4) + P(H_1T_2H_3H_4) = (\frac{1}{2})^4 + (\frac{1}{2})^4 = \frac{1}{8}$

$P_5 = P(T_1T_2T_3H_4H_5) + P(T_1H_2T_3H_4H_5) + P(H_1T_2T_3H_4H_5) = 3\cdot(\frac{1}{2})^5 = \frac{3}{32}$.

[In fact, for all r, $P_r = F_{r-1}/2^r$, where F_n denotes the n^{th} Fibonacci number!]

4. If $B_1, \dots, B_n$ is a partition of B, then $AB_1, \dots, AB_n$ is a partition of AB, so

$$P(AB) = P(AB_1) + P(AB_2) + \dots + P(AB_n)$$
$$= P(A|B_1)P(B_1) + \dots + P(A|B_n)P(B_n).$$

Divide by $P(B)$, and note that for $i = 1, \dots, n$, $B_i \subseteq B$ and hence $B_i = B_iB$.
Thus,

$$P(A|B) = \frac{P(AB)}{P(B)} = P(A|B_1)\frac{P(B_1)}{P(B)} + \dots + P(A|B_n)\frac{P(B_n)}{P(B)}$$
$$= P(A|B_1)\frac{P(B_1B)}{P(B)} + \dots + P(A|B_n)\frac{P(B_nB)}{P(B)}$$
$$= P(A|B_1)P(B_1|B) + \dots + P(A|B_n)P(B_n|B).$$

5. Let $F = \{\text{chosen die is fair}\}$, $B = \{\text{chosen die is biased}\}$, $O = \{\text{outcome is } 1\}$.

Then $P(F) = \dfrac{\#\text{ fair dice}}{\text{total }\#\text{ dice}} = \dfrac{f}{f+b}$, $P(B) = \dfrac{b}{f+b}$, $P(O|F) = \frac{1}{6}$, $P(O|B) = \frac{1}{3}$.

So
$$P(O) = P(O|F)P(F) + P(O|B)P(B) = \frac{1}{6}\cdot\frac{f}{f+b} + \frac{1}{3}\cdot\frac{b}{f+b}$$
and
$$P(F|O) = \frac{P(FO)}{P(O)} = \frac{P(O|F)P(F)}{P(O)} = \frac{\frac{1}{6}\cdot\frac{f}{f+b}}{\frac{1}{6}\cdot\frac{f}{f+b} + \frac{2}{6}\cdot\frac{b}{f+b}} = \frac{f}{f+2b}.$$

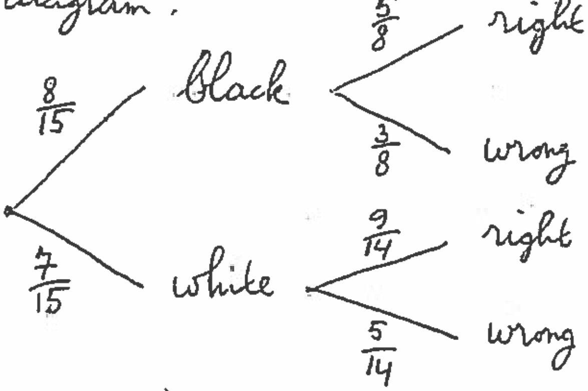
6. a) $P(\text{white}) = P(\text{white}|\text{Box }1)P(\text{Box }1) + P(\text{white}|\text{Box }2)P(\text{Box }2)$
$$= \frac{1}{3}\cdot\frac{1}{2} + \frac{3}{5}\cdot\frac{1}{2} = \frac{1}{6} + \frac{3}{10} = \frac{5}{30} + \frac{9}{30} = \frac{14}{30} = \frac{7}{15} \quad (\text{again!})$$

b) $P(\text{Box }1|\text{white}) = \dfrac{P(\text{white}|\text{Box }1)P(\text{Box }1)}{P(\text{white})} = \dfrac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{7}{15}} = \frac{1}{6}\cdot\frac{15}{7} = \frac{5}{14}$

c) $P(\text{Box }1|\text{black}) = \dfrac{P(\text{black}|\text{Box }1)P(\text{Box }1)}{P(\text{black})} = \dfrac{\frac{2}{3}\cdot\frac{1}{2}}{1-\frac{7}{15}} = \dfrac{1/3}{8/15} = \frac{5}{8}.$

d) If the ball is white, guess box 2 (correct with probability $\frac{9}{14}$) and if black, guess box 1 (correct with probability $\frac{5}{8}$).

Tree diagram:



e) $P(\text{guess correct}) = P(\text{guess correct}|\text{black})P(\text{black}) + P(\text{guess correct}|\text{white})P(\text{white})$
$$= \frac{5}{8}\cdot\frac{8}{15} + \frac{9}{14}\cdot\frac{7}{15}$$
$$= \frac{1}{3} + \frac{9}{30} = \frac{19}{30}.$$

7. Let X be the number of times 6 appears in the seven rolls. Then $X \sim \text{binomial}(7, \frac{1}{6})$.

a) $P(6 \text{ appears at most once}) = P(X\le 1) = P(X=0) + P(X=1) = \left(\frac{5}{6}\right)^7 + \binom{7}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^6.$

b) $P(\text{two 6's in first four}\,|\,\text{three 6's in the seven}) = \dfrac{P(\text{two 6's in first four \underline{and} three in the seven})}{P(\text{three 6's in the seven})}$

$$= \frac{P(\text{two 6's in first four \underline{and} one 6 in last three})}{P(\text{three 6's in the seven})} = \frac{P(\text{two 6's in four})P(\text{one 6 in three})}{P(\text{three 6's in seven})}$$

First four rolls are independent of the last three.

$$= \frac{\binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2\cdot\binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2}{\binom{7}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^4} = \frac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}}.$$

8. Let X = # of people who show up, then $X \sim \text{binomial}(n,p)$ where $n = 260$, $p = .96$. Then $\mu = np = 249.6$, $\sigma = \sqrt{np(1-p)} \approx 3.16$. Using the normal approximation with continuity correction:

$$P(\text{overbooked}) = P(X > 250) \approx 1 - \Phi\left(\frac{250.5 - 249.6}{3.16}\right) = 1 - \Phi(0.285)$$

$$\approx 1 - .6122 = .3878$$

(by interpolating in the normal table).