Answer Key (Hw. #10)

§3.2 (13) a) $E(\text{Sum}) = E(X_1 + \dots + X_{10}) = 10\,E(X_1) = 35$, where $X_1, \dots, X_{10}$ denote the individual rolls.

b) See example done in class (or Example 9, p. 172) $\longrightarrow$ 8.458

c) Using the definition: let M be the maximum of the first 5 rolls.

$$P(M=1) = P(X_1=1, \dots, X_5=1) = \left(\tfrac{1}{6}\right)^5$$

$$P(M=2) = P(M\le 2) - P(M\le 1) = \left(\tfrac{2}{6}\right)^5 - \left(\tfrac{1}{6}\right)^5$$

etc. $\longrightarrow$ General formula: $P(M=k) = P(M\le k) - P(M\le k-1) = \left(\frac{k}{6}\right)^5 - \left(\frac{k-1}{6}\right)^5$.

so

$$E(M) = 1\cdot P(M=1) + 2\cdot P(M=2) + \dots + 6\cdot P(M=6)$$

$$= \left(\tfrac{1}{6}\right)^5 + 2\left[\left(\tfrac{2}{6}\right)^5 - \left(\tfrac{1}{6}\right)^5\right] + 3\left[\left(\tfrac{3}{6}\right)^5 - \left(\tfrac{2}{6}\right)^5\right] + 4\left[\left(\tfrac{4}{6}\right)^5 - \left(\tfrac{3}{6}\right)^5\right] + 5\left[\left(\tfrac{5}{6}\right)^5 - \left(\tfrac{4}{6}\right)^5\right] + 6\left[\left(\tfrac{6}{6}\right)^5 - \left(\tfrac{5}{6}\right)^5\right]$$

$$= 6 - \left(\tfrac{1}{6}\right)^5 - \left(\tfrac{2}{6}\right)^5 - \left(\tfrac{3}{6}\right)^5 - \left(\tfrac{4}{6}\right)^5 - \left(\tfrac{5}{6}\right)^5 \doteq 5.43.$$

Other way: Let N denote the minimum of the 5 rolls.

By symmetry, X_i has the same distribution as $7 - X_i$, for each i.

Thus $M = \max\{X_1, \dots, X_5\} \sim \max\{7-X_1, \dots, 7-X_5\} = 7 - \min\{X_1, \dots, X_5\}$

$= 7 - N$, so $E(M) = 7 - E(N)$.

Using the method from Example 9, $E(N) \doteq 0.569$. $\therefore E(M) = 5.43$

d) Let $A_i := \{i^{th}$ roll is a multiple of 3$\}$, $i = 1, \dots, 10$.

Then the expected number of multiples of 3 is $P(A_1) + \dots + P(A_{10}) = 10 \times \frac{1}{3} = \frac{10}{3}$.

e) Let X be the number of faces that fail to appear, and define

$A_i := \{$face i fails to appear in the ten rolls$\}$, $i = 1, \dots, 6$.

Then $X = \#$ of A_i that occur, so

$$E(X) = P(A_1) + \dots + P(A_6)$$

$$= \left(\tfrac{5}{6}\right)^{10} + \dots + \left(\tfrac{5}{6}\right)^{10} = 6\left(\tfrac{5}{6}\right)^{10}.$$

f) Let Y be the number of faces that appear. Then $Y = 6 - X$ (with X as in part (e)), so

$$E(Y) = 6 - E(X) = 6\left[1 - \left(\tfrac{5}{6}\right)^{10}\right]$$