

Math 4610 - Review for Final Exam

SHOW ALL YOUR WORK! NO WORK=NO CREDIT!!

1. John and Jim play a sequence of games. In each game, John and Jim both toss a fair coin. If the coins land differently, the player who tossed heads is declared the winner, and the game stops. If the coins land the same, a draw is declared, and the game continues.

- What is the probability that exactly 5 games are played?
- Find the expected number of games played.

2. Five dice are rolled. Find the chance of

- exactly 4 sixes
- a full house (three of one kind, and two of a second kind)

3. I roll a fair die, and whatever number I roll, you flip a fair coin that many times. Find, to four decimal places, the chance of getting exactly three heads.

4. Let X be a random variable with density

$$f(x) = 3x^{-4}, \quad x \geq 1.$$

Find the mean μ , the median m , and the standard deviation σ of X , and verify that $|\mu - m| \leq \sigma$.

5. Let X and Y be independent geometric random variables with parameter p . Let $Z = X + Y$. Prove by a direct calculation (without referring to Bernoulli trials) that Z has a negative binomial distribution with parameters 2 and p .

6. Let X be a random variable having a uniform distribution on the interval $(0, 4)$, and let $Y := |X - 1|$.

- Give the c.d.f. of X .
- State the range of Y .
- Determine the c.d.f. of Y . (*Hint*: consider the two cases $y \leq 1$ and $y > 1$ separately.)
- Find the density function of Y .

7. Let A and B be events with $P(A) = 1/3$ and $P(B) = 1/6$, and let N be the number of events (A and/or B) that occur.

- Find $E(N)$.
- Suppose that in addition, $P(AB) = 1/12$. Find $Var(N)$. (*Hint*: compute $P(N = 1)$ and $P(N = 2)$.)

8. Suppose you roll a four-sided die until you roll a number you have rolled before. Let

$$R_i = \{\text{the first duplicate occurs on the } i\text{-th roll}\}, \quad i = 1, 2, 3, \dots$$
$$D_i = \{\text{the first } i \text{ rolls show different numbers}\}, \quad i = 1, 2, 3, \dots$$

a) Draw a suitable tree diagram for the above events, and label each branch with the corresponding (conditional) probability.

b) Find $P(R_i)$, for $i = 1, 2, 3, 4, 5$, and 6.

9. A freight train carries containers for 20 different customers. Each customer independently supplies a random number of containers that is Poisson distributed with parameter $\mu = 4$.

Let N be the total number of containers on the train.

a) Write a formula for $P(N = k)$. Specify where the formula is valid.

b) Find $E(N)$ and $\text{Var}(N)$.

c) Use the normal approximation to find the probability that the train carries more than 100 containers.

10. Let X and Y be independent random variables that are exponentially distributed with parameters λ and μ , respectively. Let $T := \min(X, Y)$.

Prove that the distribution of T is exponential with parameter $\lambda + \mu$. Show ALL of the details! (*Hint*: Consider the probability $P(T > t)$.)

11. A test is repeated until done satisfactorily. Assume that repetitions of the test are independent and each has probability 0.25 of being satisfactory. The first 5 tests cost \$100 each, and each test thereafter costs \$40. Find the expected cost of the tests until a satisfactory result is obtained. (*Hint*: introduce appropriate random variables.)

12. Extra credit!!

Let A and B be events with $P(A) = p_1$ and $P(B) = p_2$. Assume $p_1 + p_2 > 1$. Assume also that $p_1 > p_2$. What are the largest and smallest that $P(AB)$ could possibly be? Explain.