Math 4610 Review for Final Exam, Solutions

1. For a single game, $P(\text{someone wins}) = P(HT) + P(TH) = \frac{1}{2}$, so the total number of games played has a geometric distribution with $p = \frac{1}{2}$. Thus,
   a) $P(5 \text{ games}) = (1-p)^4 p = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$ b) Exp. # of games $= \frac{1}{p} = 2$.

2. a) $P(4 \text{ sixes}) = \binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)$ b) $P(\text{full house}) = 6 \cdot 5 \cdot \binom{5}{3}\left(\frac{1}{6}\right)^5$
   ($6 \cdot 5$ ways to choose the two numbers, $\binom{5}{3}$ distinct arrangements, each with prob. $\left(\frac{1}{6}\right)^5$).

3. ~~Let N = # of customers who show up. Then $N \sim Bin(n,p)$ with $n = 320$, $p = 0.9$. Then $\mu = np = 288$, $\sigma = \sqrt{npq} \approx 5.37$. So~~

Solution of #3 at bottom of next page (after #12)

   ~~$P(N \geq 301) \approx 1 - \Phi\left(\frac{300.5 - 288}{5.37}\right) = 1 - \Phi(2.33) = 1 - 0.9901 = 0.0099$.~~

4. (i) $\mu = \int_1^\infty x \cdot 3x^{-4}\,dx = 3\int_1^\infty x^{-3}\,dx = -\frac{3}{2}x^{-2}\Big|_1^\infty = \frac{3}{2}$

   (ii) $F(x) = \int_1^x f(u)\,du = 1 - \int_x^\infty 3u^{-4}\,du = 1 - (-u^{-3})\Big|_x^\infty = 1 - x^{-3}$

   $F(x) = \frac{1}{2} \iff x^{-3} = \frac{1}{2} \iff x^3 = 2 \iff x = \sqrt[3]{2}$. Thus $m = \sqrt[3]{2}$.

   (iii) $E(X^2) = \int_1^\infty x^2 \cdot 3x^{-4}\,dx = 3\int_1^\infty x^{-2}\,dx = -3x^{-1}\Big|_1^\infty = 3$, so $Var(X) = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$
   and $SD(X) = \frac{\sqrt{3}}{2} = \sigma$.

   (iv) Note that $|\mu - m| = \left|\frac{3}{2} - \sqrt[3]{2}\right| = .240 < .866 = \frac{\sqrt{3}}{2} = \sigma$.

5. $Range(Z) = \{2,3,4,\ldots\}$ and for n in this range, $P(Z=n) = P(X+Y=n) =$
   $= \sum_{k=1}^{n-1} P(X=k, Y=n-k) = \sum_{k=1}^{n-1} P(X=k)P(Y=n-k) = \sum_{k=1}^{n-1} p(1-p)^{k-1}p(1-p)^{n-k-1}$
   $= p^2 \sum_{k=1}^{n-1}(1-p)^{n-2} = (n-1)p^2(1-p)^{n-2} = \binom{n-1}{2-1}p^2(1-p)^{n-2}$, so $Z \sim neg.\,bin(2,p)$.

6. a) $F_X(x) = \frac{x}{4}$ for $0 \leq x \leq 4$, $F_X(x) = 0$ for $x < 0$, $F_X(x) = 1$ for $x > 4$
   b) $Range(Y) = [0,3)$.
   c) For $0 \leq y \leq 1$, $F_Y(y) = P(Y \leq y) = P(|X-1| \leq y) = P(-y \leq X-1 \leq y)$
   $= P(1-y \leq X \leq 1+y) = \frac{1+y-(1-y)}{4} = \frac{y}{2}$.
   For $1 < y \leq 3$, $F_Y(y) = P(1-y \leq X \leq 1+y) = P(0 \leq X \leq 1+y)$ (since $1-y \leq 0$)
   $= \frac{1+y}{4}$

   Thus, $F_Y(y) = \begin{cases} 0, & y < 0 \\ y/2, & 0 \leq y \leq 1 \\ (1+y)/4, & 1 < y \leq 3 \\ 1, & y > 3. \end{cases}$

   d) $f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{2}, & 0 < y < 1 \\ \frac{1}{4}, & 1 < y < 3 \\ 0, & \text{elsewhere.} \end{cases}$ (We leave $f_Y(y)$ undefined for $y = 1$)

7. a) $N = I_A + I_B$, so $E(N) = P(A) + P(B) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$.

b) $P(N=2) = P(AB) = \frac{1}{12}$, $P(N=1) = P(AB^c) + P(A^c B) = [P(A) - P(AB)] + [P(B) - P(AB)]$
$= \frac{1}{3} + \frac{1}{6} - 2 \cdot \frac{1}{12} = \frac{1}{3}$. So $E(N^2) = 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{12} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$, $Var(N) = \frac{2}{3} - (\frac{1}{2})^2 = \frac{5}{12}$.

8. a)
$$\begin{tikzcd}
 & R_1 & R_2 & R_3 & R_4 & R_5 & R_6 & R_7 \\
{} \arrow[ur, "0"] \arrow[r, "1"'] & D_1 \arrow[ur, "1/6"] \arrow[r, "5/6"'] & D_2 \arrow[ur, "2/6"] \arrow[r, "4/6"'] & D_3 \arrow[ur, "3/6"] \arrow[r, "3/6"'] & D_4 \arrow[ur, "4/6"] \arrow[r, "2/6"'] & D_5 \arrow[ur, "5/6"] \arrow[r, "1/6"'] & D_6 \arrow[ur, "1"] \arrow[r, "0"'] & D_7
\end{tikzcd}$$

b) $P(R_1) = 0$, $P(R_2) = P(D_1) P(R_2 | D_1) = \frac{1}{6}$, $P(R_3) = P(D_1 D_2) P(R_3 | D_1 D_2) = \frac{5}{6} \cdot \frac{2}{6}$,
$P(R_4) = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6}$, $P(R_5) = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{4}{6}$, $P(R_6) = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{5}{6}$, $P(R_7) = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{1}{6} \cdot 1$

9. a) $N \sim Ps(80)$ so $P(N=k) = e^{-80} \frac{80^k}{k!}$, $k = 0, 1, 2, \ldots$

b) $E(N) = Var(N) = 80$

c) Since $N = N_1 + N_2 + \ldots + N_{20}$ where the N_i's are independent and have the same ($Ps(4)$) distribution, the CLT applies. Thus

$$P(N > 100) \approx 1 - \Phi\left(\frac{100.5 - 80}{\sqrt{80}}\right) = 1 - \Phi(2.29) = 1 - .9890 = .0110$$

10. See Example 3, p. 317.

11. Let T = number of repetitions needed, C = total cost. Then $T \sim$ geometric$(\frac{1}{4})$ and

$$C = \begin{cases} 100T, & \text{if } T \le 5 \\ 500 + 40(T-5) = 40T + 300, & \text{if } T > 5. \end{cases} =: g(T)$$

Then
$$E(C) = \sum_{k=1}^{\infty} g(k) P(T=k) = \sum_{k=1}^{5} 100k \cdot \left(\tfrac{3}{4}\right)^{k-1}\left(\tfrac{1}{4}\right) + \sum_{k=6}^{\infty} (40k + 300)\left(\tfrac{3}{4}\right)^{k-1}\left(\tfrac{1}{4}\right)$$
$$= 25\left(1 + 2 \cdot \tfrac{3}{4} + 3 \cdot \left(\tfrac{3}{4}\right)^2 + 4 \cdot \left(\tfrac{3}{4}\right)^3 + 5 \cdot \left(\tfrac{3}{4}\right)^4\right) + 40 \sum_{k=6}^{\infty} k P(T=k) + 300 P(T > 5)$$
$$= \frac{47725}{256} + 40\left(E(T) - \sum_{k=1}^{5} k P(T=k)\right) + 300 \cdot \left(\tfrac{3}{4}\right)^5$$
$$= \frac{47725}{256} + 40\left(4 - \frac{1909}{1024}\right) + 300\left(\tfrac{3}{4}\right)^5 = \frac{21955}{64} \approx \$343.05$$

12. $P(AB)$ is largest when $B \subseteq A$ (this is possible since $P(B) = p_2 < p_1 = P(A)$); and then $P(AB) = P(B) = p_2$. On the other hand, $P(AB)$ is smallest when A and B overlap as little as possible, i.e. when $A \cup B = \Omega$. Then $P(AB) = P(A) + P(B) - P(A \cup B)$
$= p_1 + p_2 - 1$. (Note that $A \cup B = \Omega$ is possible since $p_1 + p_2 > 1$)

3. Let X = the number rolled with the die. Then
$$P(3 \text{ heads}) = \sum_{n=1}^{6} P(3 \text{ heads} | X = n) P(X = n) = \sum_{n=3}^{6} \binom{n}{3} \left(\tfrac{1}{2}\right)^n \cdot \frac{1}{6} = \frac{1}{6},$$
as given $X = n$, the number of heads has a binomial $(n, \frac{1}{2})$ distribution.